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An Historical and Modern View on Bell's Inequality

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Abstract

After a general introduction, we introduce the reader to some subtleties of quantum mechanics from an historical perspective. We discuss the famous EPR paper, in which Einstein, Podolsky, and Rosen claim that either we live in a world with 'spooky actions at a distance' or quantum mechanics is incomplete. After discussing various completeness 'proofs', a lot of attention will be paid to Bell's celebrated inequality. John Bell provided – for the first time in history – an experimentally verifiable criterion, deciding which of the two EPR choices is the correct one. It turned out that the notion of 'local realism' must be rejected. This makes most physicists believe that we live in a non-local universe.[1] We also study superluminal signaling and superluminal causation. In the final part of this work, we discuss the modern mathematical C*-algebraic framework, thereby focusing on the rôle of commutativity as a criterion for satisfying Bell's inequality. A short guide through the literature is provided at the very end.

[1]Although non-locality seems weird, it may well be the way Nature is. To quote Richard Feynman (Int. J. Th. Phys. **21**,467, 1982): "It has not yet become obvious to me that there is no real problem. I cannot define the real problem, therefore I suspect there's no real problem, but I am not sure there is no real problem. So that's why I like to investigate things."

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1 General introduction

The paper of Einstein, Podolsky, and Rosen (EPR) [14] pointed out a very counterintuitive and astonishing aspect of quantum theory: the *non-separability* of two widely distant and non-interacting quantum systems that had interacted in the past. EPR conclude that either (a) the wave function does not provide a complete description of physical reality, or (b) their criterion of *local realism* contradicts quantum mechanics.

It was John Bell who brought this initially philosophical issue down to empirical science by providing experimentally falsifiable implications of so called local hidden variable theories. This was a major turning point in the EPR discussion and will be the focal point of this paper.

We start by discussing some aspects of quantum theory, like the concept of states, the measurement problem, interpretations of the theory etcetera. Then we come to the prehistory of Bell's inequality: the EPR paradox, von Neumann's completeness proof, and the Kochen-Specker theorem. Via causality, common causes, and local hidden variables, we slowly move on to Bell's theorem. In chapter three we study Bell's inequality in some depth, paying attention to both its assumptions and its derivation, but also to generalized versions of it, like the Clauser-Horne inequalities. Chapter four is about special relativity and the implications of violations of Bell's inequality for the concept of causality. In chapter five we proceed by discussing how Bell's inequality *could* be, and *has* been, tested experimentally. Finally the discussion takes on a more mathematical form; we look at the modern C*-algebraic framework. This approach puts Bell's inequality in a different perspective.

2 Some aspects of quantum theory

In this section we discuss some aspects of quantum theory. This will be mainly done from an historical perspective, thereby providing the reader with some background to Bell's discovery.

2.1 Formalism and interpretations

The formalism of quantum mechanics is nowadays firmly established. The interpretation of this formalism, however, is still a point of discussion. There are three main schools of thought regarding quantum theory: (1) *realist*, (2) *orthodox* (or *Copenhagen*), and (3) *agnostic*. They can be characterized by looking at the answers they would give to the following question: suppose we measure the position of a particle, and find it to be at point x_0; where was the particle just before the measurement?

The realist position is that the particle was at x_0 (it should be noted that if this were the case quantum mechanics would be an incomplete theory). The orthodox position is that the particle wasn't really anywhere. One can also refuse to answer, saying that the question is a metaphysical one; this is the agnostic position. In the course of time it became clear (especially due to Bell and Aspect) that the orthodox school had been correct. A particle simply does not have a precise position before we measure it. In a sense our measurement *creates* the result! Observations not only disturb what is to be measured, they produce it. The reader may want to take a look at Mermin's "Is the moon there when nobody looks?" [31], which is a nice discussion (at an elementary level) on reality and the quantum theory.

We will not investigate the question of what it *means* to interpret the formalism of quantum mechanics. This is a difficult question, and as Max Jammer put's it in [25]: "In fact, just as physicists disagree on what *is* the correct interpretation on quantum mechanics, philosophers of science disagree on what it *means* to interpret such a theory."

In our discussion we will need some mathematical formalism, which will be introduced gradually throughout this paper. The elements of a Hilbert space $\mathcal{H}$ are vectors $\phi, \psi, \ldots$, and their inner product is denoted by $\langle \phi, \psi \rangle$. A linear operator A is called self-adjoint if $\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle$ for all ϕ, ψ in $\mathcal{H}$. From the spectral theorem of functional analysis we know that to every self-adjoint linear operator A corresponds a unique resolution of identity, that is, a set of 'projections' $E^{(A)}(\lambda)$ or briefly E_λ, $(\lambda \in \mathbb{R})$, such that (1) $E_\lambda \leq E_{\lambda'}$ for $\lambda \leq \lambda'$, (2) $E^{-\infty} = 0$, (3) $E^{\infty} = I$, (4) $E_{\lambda+0} = E_\lambda$, (5) $I = \int_{-\infty}^{\infty} dE_\lambda$, (6) $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$, and finally (7) for all λ, E_λ commutes with any operator that commutes with A. The spectrum of A is the set of all λ that are not in the interval in which E_λ is constant.

We will use some primitive notions in our discussion, namely, the notions of a *system*, a *physical quantity* (or *observable*), and a *state*; in addition the notions of *probability* and *measurement* are used without interpretation. Having said this, we are ready to introduce the formalism of quantum mechanics following Dirac and von Neumann.

1. To every pure state[2] system corresponds a Hilbert space $\mathcal{H}$ whose vectors (state vectors, wave functions) completely describe the pure states of the system.

2. To every observable $\mathfrak{A}$ corresponds uniquely a self-adjoint operator A acting in $\mathcal{H}$.

3. For a system in state ϕ, the probability that the result of a measurement of the observable $\mathfrak{A}$, represented by A, lies between λ_1 and λ_2 is given by $\|(E_{\lambda_2} - E_{\lambda_1})\phi\|^2$, where E_λ is the resolution of the identity defined by A.

4. The time development of the state vector ϕ is determined by the Schrödinger equation $H\phi = i\hbar\frac{\partial\phi}{\partial t}$, where the Hamiltonian H is the evolution operator and $\hbar$ is Planck's constant divided by 2π.

5. Define $P_{\lambda_1,\lambda_2} = E_{\lambda_2} - E_{\lambda_1}$. (i) If $P_{\lambda_1,\lambda_2}\phi = 1$ than the observable A of the system in state ϕ has with certainty a value in the interval $[\lambda_1, \lambda_2]$. (For example after a measurement.) (ii) If ϕ is no eigenstate of some P_{λ_1,λ_2} than A in state ϕ is *undetermined*.

6. If a measurement of the observable $\mathfrak{A}$, represented by A, yields a result between λ_1 and λ_2, then the state of the system immediately after the measurement is an eigenfunction of $E_{\lambda_2} - E_{\lambda_1}$.

Axioms 1 and 2 associate the primitive notions with mathematical entities. Axiom 3 plays a crucial role for all questions of interpretation. It establishes a connection between the physical data and mathematics. From axiom 3 it follows that the result of a measurement of an observable $\mathfrak{A}$, represented by A, is an element of the spectrum of A. Axiom 3 can be regarded as the axiom of 'quantum statistics'. Axiom 4 is the axiom of 'quantum dynamics'. Axiom 6 is called the *projection postulate* (also: the *collapse of the wave function*) and is more controversial than the rest. It states that in the discrete case, immediately after having obtained the eigenvalue λ_j of A when measuring $\mathfrak{A}$, the state of the system is an eigenvector of P_j, the projection on the eigenvector belonging to λ_j. It has been rejected by some theorists on grounds to be discussed in due course. Axiom 5 has to be added because from axiom 6 only it is not clear that A is determined in that case.

2.1.1 Pure states, mixed states, and density operators

A density matrix, or density operator, is used in quantum theory to describe the statistical state of a quantum system. It is the quantum-mechanical analogue to a phase-space density (probability distribution of position and momentum) in classical statistical mechanics. The need for a statistical description via density matrices arises because it is not possible to describe a quantum mechanical system that undergoes general quantum operations such as measurement, using

[2]One can make (or better said *has* to make) a distinction between two kinds of states; pure states, and mixed states. We will come back to this issue in the following section.

exclusively states represented by ket vectors. In general a system is said to be in a mixed state, except in the case that the state is not reducible to a convex combination of other statistical states; in that case it is said to be in a pure state.

A typical situation in which a density matrix is needed is an entanglement between two subsystems, where each individual system must be described by a density matrix even though the complete system may be in a pure state. Formally, a density operator (or density matrix) ρ is an operator acting on H, the Hilbert space of the system in question.

In general, for a mixed state, where the system is in the quantum-mechanical state $|\psi_j\rangle$ with probability p_j, the density matrix is the sum of the projectors, weighted with the appropriate probabilities

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|. \tag{2.1}$$

The density matrix is used to calculate the expectation value of any operator A of the system, averaged over the different states ψ_j. This is done by taking the trace of the product of ρ and A:

$$\langle A\rangle = \mathrm{tr}(\rho A) = \sum_j p_j \langle\psi_j|A|\psi_j\rangle. \quad \text{(mixed state)} \tag{2.2}$$

A pure state is a special state for which one of the p_j equals 1. Note that, calling the wave function corresponding to this j simply ψ, we have for pure states

$$\mathrm{tr}(\rho A) = \langle\psi|A|\psi\rangle. \quad \text{(pure state)} \tag{2.3}$$

2.2 The measurement problem

Before we start our journey through the history of quantum mechanics and no-hidden-variable theories we first address the measurement problem. What exactly is *a measurement*? And when is the projection postulate supposed to take over from the Schrödinger dynamics? There are two ways a state can change. In general, states change via axiom 4, that is via the Schrödinger equation. However, when a measurement occurs, states change via the projection postulate[3] (axiom 6). Although it is perhaps not entirely satisfactory, the following 'definition' of a measurement looks acceptable: A measurement occurs at the moment when a microscopic system (described by quantum mechanics) interacts with a macroscopic system (described by classical mechanics) in such a way as to leave a record. By taking a closer look at axiom 1 and 2 we see that they are quite different. The first one is deterministic. For any initial state $|\Psi(t_1)\rangle$, there is exactly one final state $|\Psi(t_2)\rangle$ that it evolves to. The second one, however, is *in*deterministic. When a measurement of B is made on a state $|\Psi\rangle = \sum a_i|b_i\rangle$ expanded in the eigenvector

[3]This is *not* what is known as von Neumann's projection postulate, where pure states transit into mixed states.

basis of B with result b_i, then $|\Psi\rangle$ collapses to $|b_i\rangle$ with probability $|a_i|^2$. Also this 'collapse' happens instantaneously. There have been attempts to experimentally prove the collapse of the wave function. A famous result is the quantum Zeno effect. Misra and Sudarshan [33] proposed to take an unstable system and subject it to repeated measurements. Each observation makes the wave function collapse, resetting the clock, and so it is possible to delay the expected transition to the lower state. They conclude that a continuously observed system never decays at all. However a continuous observation is impossible in practice: for example, a particle in a bubble chamber is only at some specific times interacting with the atoms in the chamber, and for the quantum Zeno effect the successive measurements must be made extremely rapidly. In fact, no experiments have yet provided as compelling a confirmation of the collapse of the wave function as its designers hoped. The necessity of the collapse of the wave function is therefore purely of theoretical nature.

2.3 Early interpretations and the indeterminacy relations

The earliest consistent (partial) theory of quantum phenomena was Heisenberg's matrix mechanics (1925)[4], in which physical quantities were represented by sets of time-dependent complex numbers [23]. A few months later, Schrödinger introduced his wave mechanical formalism, including his well known time-dependent and time-independent wave equations. As later clarified by von Neumann, Schrödinger [39] discovered that his own formalism and Heisenberg's matrix calculus are mathematically equivalent despite the many apparent dissimilarities. Heisenberg used of the sequence space l^2, the set of all infinite sequences of complex numbers whose squared absolute values give a finite sum, whereas Schrödinger used the space $L^2(-\infty, +\infty)$ of all complex-valued square-summable (Lebesgue) measurable functions. It was realized (by von Neumann) that there exists a 1-1 correspondence between the 'wave functions' in L^2 and the 'sequences' of complex numbers in l^2, between Hermitian differential operators and Hermitian matrices. Solving the eigenvalue problem of an operator in L^2 is equivalent to diagonalizing the corresponding matrix in l^2.

Schrödinger defined the function ψ as satisfying the (mysterious) wave equation

$$\frac{4\pi m}{ih}\frac{\partial\psi}{\partial t} = \frac{8\pi^2 mV}{h^2}\psi - \nabla^2\psi, \tag{2.4}$$

where $\psi = \psi(x, y, z, t)$ for a one-particle system or $\psi = \psi(x_1, \ldots, z_n, t)$ for a system of n particles. He interpreted quantum theory as a classical theory of waves. In his view, physical reality consists of waves and of waves only. He denied the existence of discrete energy levels and quantum jumps, on the ground that in wave mechanics the discrete eigenvalues are eigenfrequencies of waves rather than energies. He wrote [37] (and he never changed his view on

[4] It was Heisenberg's insight that quantum mechanical variables do not commute, that formed the basis of the creation of matrix mechanics by Born, Jordan, and Heisenberg himself.

this point) to Max Planck on May 31, 1926: "The concept 'energy' is something that we have derived from macroscopic experience and really *only* from macroscopic experience. I do not believe that it can be taken over into micro-physics just like that, so that one may speak of the energy of a single particle oscillation. The energetic property of the individual particle is *its frequency*." More on Schrödinger's so called 'electromagnetic interpretation' and also on the so called 'hydrodynamic interpretation', of which the earliest version [5] was formulated by E. Madelung, can be found in [25]. We will now turn to the (history of the) *indeterminacy relations*.

Heisenberg was concerned with the following questions: (1) Does the formalism allow for the fact that the position of a particle and its velocity are determinate at a given moment only with limited degree of precision? (2) Would such imprecision, if admitted by the theory, be compatible with the optimum of accuracy obtainable in experimental measurements?

Indeed Heisenberg answers (1) positively, and finds for a Gaussian distribution $\Delta q \Delta p = h/4\pi$, where q is the position coordinate, p the momentum, and Δ gives the standard deviation. To answer the second question Heisenberg analysed the so called "gamma-ray microscope experiment." Not going into details we note that Heisenberg had the operational view that one had to refer to a definite experiment by which "the position" is to be determined; otherwise the concept has no meaning. He saw indeterminacy as a limitation of the applicability of classical notions, like position or momentum, to microphysical phenomena.

It was Robertson, who proved for the first time that the product of the standard deviation of two self-adjoint operators A and B is never less than half the absolute value of the mean of their commutator $C = i[A, B]$. The clue of the proof is to write down a suitable expression and to look at the discriminant; one finds the well known $\Delta A \Delta B \geq \frac{1}{2}|\langle AB - BA \rangle|$. As Ditchburn and Sygnge proved, *equality* holds if and only if the scatter is a Gaussian distribution (as assumed by Heisenberg).

An interesting philosophical implication that Heisenberg drew from these relations was that it is not possible to have exact knowledge of the present, so that, identifying the law of causality with the statement that "exact knowledge of the present allows the future to be calculated" stops making sense in quantum mechanics. Note that this does *not* mean that 'causality is proven wrong'. If a premise of an implication is false, this does not entail the falsity of the implication itself (and the implication alone is the causality law).

2.4 The early complementarity interpretation

The term "complementarity" was first introduced by Niels Bohr in his Como lecture: "On one hand, the definition of the state of a physical system, as ordinarily understood, claims the elimination of all external disturbances. But in that case, according to the quantum postulate, any

[5] Attempts to interpret quantum mechanics in terms of hydrodynamical models were not only confined to the early stages of the theory.

observation will be impossible, and, above all, the concepts of space and time lose their immediate sense. On the other hand, if in order to make observation possible we permit certain interactions with suitable agencies of measurement, not belonging to the system, an unambiguous definition of the state of the system is naturally no longer possible, and there can be no question of causality in the ordinary sense of the word. The very nature of quantum theory thus forces us to regard the space-time coordination and the claim of causality, the union of which characterizes the classical theories, as complementary but exclusive features of the description, symbolizing the idealization of observation and definition respectively."

It is not easy to give a clear-cut definition of what Bohr meant when he spoke about "complementarity" in atomic physics. Einstein complained, in 1949, that "despite much effort which I have expended on it, I have been unable to achieve a sharp formulation of Bohr's principle of complementarity." However, it is clear that above all, Bohr referred to the impossibility of carrying out a causal description of atomic phenomena which, at the same time, is also a space-time description. Defining Bohr's notion of complementarity is not easy, but the notion of *complementarity interpretation* seems to raise fewer difficulties.

A given theory T admits a complementary interpretation if the following conditions are satisfied: (1) T contains (at least) two descriptions D_1 and D_2 of its substance-matter; (2) D_1 and D_2 refer to the same universe of discourse U (in Bohr's case, microphysics); (3) neither D_1 nor D_2, if taken alone, accounts exhaustively for all phenomena of U; (4) D_1 and D_2 are mutually exclusive in the sense that their combination into single description would lead to logical contradictions. (5) D_1 and D_2 together provide a full description of T.

2.5 The Bohr-Einstein debate

(This section can be skipped without affecting the continuity.) The Bohr-Einstein debate is often seen as one of the greatest intellectual debates of the last century. Einstein was convinced that the world exists 'out there' independent of us; Bohr on the other hand thought that the world 'out there' is not something that enjoys an independence of its own, but is inextricably tied up with our perceptions of it. Since it is not on the main road of this paper, we will keep our discussion limited and discuss only Einstein's photon box experiment. At the Sixth Solvay Conference (1930) Einstein presented the following argument to refute Heisenberg's relation $\Delta E \Delta t > \hbar/2$. The reader should note that this relation is of another type than the momentum-position one: the energy-time relation does not follow directly from the formalism. However it seems that in a relativistic theory the *Et*-relation would be a consequence of the *px*-relation (think of the four vectors (t,**x**) and (E,**p**)).

Consider the following experiment. We have a box with ideally reflecting walls, filled with radiation and equipped with a shutter, which is operated by a clockwork enclosed in the box. Assume that the clock is set to open the shutter at time $t = t_1$ for a short interval $t_2 - t_1$ so that a single photon can be released. By weighing the whole box before and after the emission of the accurately timed radiative pulse of energy, the difference in energy content of the box could, as Einstein pointed out, be determined with an arbitrarily small error ΔE from the mass-energy relation $E = mc^2$. From the principle of energy conservation the difference in energy content of the box is exactly equal to the energy of the emitted photon. Thus in this way we can predict the energy of the photon and the time of its arrival at a distant screen with arbitrarily small indeterminacies ΔE and Δt in contradiction to Heisenberg's time-energy relation.

Ironically Bohr countered this argument with Einstein's own general theory of relativity. The early next morning at the Solvay Congress, after what was for sure a sleepless night, Bohr said roughly the following. (To counter Einstein's challenge Bohr made use of the relativistic red-shift formula

$$\Delta T = T\frac{\Delta\phi}{c^2}, \tag{2.5}$$

which expresses the change of rate ΔT during a time interval T for a clock placed in a gravitational field through a potential difference $\Delta\phi$.)

Suppose that the box is suspended in a spring-balance and is furnished with a pointer to read its position on a scale fixed to the balance support.

By adjusting the balance to its zero position (by means of suitable loads) the weighing of the box may be performed with any given accuracy Δm. Any determination of the position of the pointer with a given accuracy Δx will involve a minimum Δp for the momentum of the box, with $\Delta x \Delta p \approx h$. Now Bohr claimed that we have

$$\Delta p \approx \frac{h}{\Delta q} < Tg\Delta m, \tag{2.6}$$

since Δp must be smaller than the total momentum which, during the whole interval T of the balancing procedure, can be given by the gravitational field to a body with a mass Δm. By comparing 2.5 and 2.6 we see that after the weighing procedure there will be a latitude in our knowledge of the adjustment of the clock given by

$$\Delta T > \frac{h}{c^2\Delta m} \text{ or } \Delta T\Delta E > h, \tag{2.7}$$

in accordance with the indeterminacy principle. Einstein accepted Bohr's counterexample and gave up any hope of refuting the quantum theory on the grounds of internal inconsistency. As we shall see in the following section, he concentrated on demonstrating the incompleteness of quantum mechanics.

2.6 The Einstein-Podolsky-Rosen (EPR) paradox

The EPR paradox (it is called a paradox, since it conflicts with our classical intuition – specifically, with the principle of locality) draws attention to a phenomenon predicted by quantum mechanics known as quantum entanglement (some call it ‘quantum weirdness’), in which measurements on spatially separated quantum systems can instantaneously influence one another. As a result, quantum mechanics violates the principle of locality, that states that changes performed on one physical system should have no immediate effect on another spatially separated system.

The principle of locality *seems* very acceptable, both on intuitive grounds and because at first sight it seems to be a natural outgrowth of the theory of special relativity. According to special relativity, information can never be transmitted faster than the speed of light, or causality would be violated. Now the point is, that quantum mechanics violates locality without violating causality, because no information can be transmitted using entanglement.

Einstein, Podolsky and Rosen were unwilling to abandon locality. They suggested that quantum mechanics is not a complete theory, just an (admittedly successful) statistical approximation to some yet-undiscovered description of Nature.

The difficulties arises from the fact that quantum mechanics treats *two* particles, which interacted in the past (and so became entangled) and flew apart, as *one* object. When one such particle ‘is changed’, the other will change too, instantly. Einstein called this behaviour ‘spooky action at a distance’, and considered it unacceptable. Before most physicists accepted it as real and inevitable, one escape route had to be closed, namely the possible existence of ‘hidden parameters’. John Bell has closed that escape route.

2.6.1 A closer investigation

We will now start with a short summary of the paper, that Einstein, Podolsky, and Rosen published in 1935, [14]. This probably will raise more questions than it will provide answers, but hopefully a lot will become clear in due course. Keep in mind that the important part is ‘in the locality’ and not so much ‘in the elements of reality’.

In the words of Einstein, Podolsky, and Rosen: “in a complete theory there is an element corresponding to each element of reality. A sufficient condition for the reality of a physical quantity is the possibility of predicting it with certainty, without disturbing the system.” What we know from quantum mechanics is that in the case of two physical quantities described by noncommuting operators, the knowledge of one precludes the knowledge of the other; EPR claim that therefore “either (1) the description of reality given by the wave function in quantum mechanics is not complete or (2) these two quantities cannot have simultaneous reality.” They could perhaps both be true, but they cannot both be false. EPR show that if (1) is false then (2) is also false. “Therefore one is thus led to conclude that the description of reality as given by the wave function is not complete.”

Note that EPR only question the *completeness* of quantum mechanics, not the *correctness*. In their article they describe an example that is meant to show that quantum mechanics is an *in*complete theory.

EPR start of defining the exact meaning of completeness. They state a necessary condition for a complete physical theory: every element of the physical reality must have a counterpart in the physical theory. So as soon as we find what these “elements of the physical reality” are, we will be done: we will be able to see whether our theory is complete or not.

EPR propose a sufficient condition for this physical **reality**: if, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

We will now look at where locality comes in. Consider two separate systems I and II that interacted with each other from say $t = 0$ to $t = T$. With Schrödinger’s equation we can (if we know what the state was at some point in time) calculate the state of the combined system for all $t > 0$ (so in particular for $t > T$).

EPR assume that their *reality* criterion is totally reasonable, so they take it as a given fact. By measuring certain quantities in system I we can deduce *with certainty* certain values of quantities in system II, without, and this is important, in any way disturbing system II. This is also an assumption. It is called the **locality** assumption. The reader should note that it does not have to be that way at all. It can very well be that in some way these two systems are connected in a way (remember that the wave function that describes everything is the same function for both systems). These two assumptions, of ‘reality’ and of ‘locality’, lead us to the conclusion that (1) is true, since (2) is false with our assumptions: our quantities *do* have simultaneous reality as defined above, since by measuring quantities in system I we can deduce what the value of these quantities are in system II without disturbing system II. Conclusion: EPR show that acceptance of (in their eyes very reasonable) local realism leads us to incompleteness of quantum mechanics.

At this point we can only say that in the decades after the EPR paper it turned out that quantum mechanics is indeed a complete theory. Therefore the combination ‘reality’ + ‘locality’ is not a valid assumption since it leads to a contradiction.

2.7 Von Neumann’s completeness proof

Before coming back to EPR and the developments after the EPR paper, we will look at John von Neumann’s completeness proof. Von Neumann was mainly attracted by the problem of determining the precise nature of the *statistical character* of quantum mechanics. Why can only statistical statements be made about the values of the observables involved? He attempted to ground the completeness of quantum mechanics in a different way from Bohr and Heisenberg. Von Neumann saw the problem of completeness of quantum mechanics as that of proving that no hidden variables can exist on the basis of certain structural features of quantum mechanics, whereas Bohr and Heisenberg were concerned with the peculiarities of measurement at the micro-level. The fact that ensembles described by the same state function exhibit dispersion, suggested to von Neumann two possible interpretations of a statistical theory (see [35], p. 302):

Case I The individual systems $s_1, ..., s_N$ of the ensemble may be in different states, so that the ensemble $[s_1, ..., s_N]$ is defined by their relative frequencies. In this case the fact that we do not obtain sharp values for the physical quantities is caused by our lack of information: we do not know in which state we are measuring, and therefore we cannot predict the results.

Case II All individual systems $s_1, ..., s_N$ are in the same state, but the laws of Nature are not causal. Then the cause of the dispersions is not our lack of information, but Nature itself.

Before we continue, a short intermezzo: what exactly is meant by the word *dispersion*? Let A be a *random variable*. This is a quantity that can assume different values according to the outcome of an experiment. A measure of the statistical 'scatter' of A about the expectation value is given by the *variance*

$$(\Delta A)^2 = \langle A - \langle A \rangle^2 \rangle,$$

where the brackets give the expectation value ($\langle A \rangle := \int A \, \mathrm{d}p$). The quantity ΔA is called the dispersion of A.

Loosely speaking, in Case I the absence of sharp values for physical quantities is caused by *our lack of information,* whereas in Case II it is *Nature itself*, that causes the statistical scatter. We now come to (the structure of) von Neumann's completeness proof, which may seem a bit vague since we did not yet introduce notions like 'homogeneous ensemble'. Here is an outline of the proof: *If we do not obtain sharp values for the physical quantities because of our lack of information (Case I), then no dispersive ensemble can be homogeneous. But every ensemble is dispersive; so no ensemble could be homogenous. But since homogeneous ensembles do exist in quantum mechanics (soon to be discussed), the assumption concerning the existence of hidden variables has been refuted.* Therefore, von Neumann was in favor of Case II. We will now study the proof in more detail. Von Neumann's assumptions are the following:

1. If a quantity (observable) $\mathcal{R}$ is represented by the operator R, then a function f of this observable is represented by the operator $f(R)$ and if $\mathcal{R}$ is "by nature" nonnegative, then its expectation value $\langle \mathcal{R} \rangle$ is nonnegative.

2. If quantities are represented by the operators $R, S, \ldots$, then the sum of these quantities is represented by the operator $R + S + \cdots$, regardless of whether the operators commute or not.

3. If $\mathcal{R}, \mathcal{S}, \ldots$ are arbitrary quantities and $a, b, \ldots,$ real numbers, then $\langle a\mathcal{R} + b\mathcal{S} + \cdots \rangle = a\langle \mathcal{R} \rangle + b\langle \mathcal{S} \rangle + \cdots$.

The first assumption could also have been, which is the same, that for all self-adjoint R (so $R = R^\dagger$) we have $\langle R^2 \rangle \geq 0$. We will now reproduce von Neumann's proof that no quantum mechanical ensemble is dispersion-free (which means that $\Delta A = 0$).

Von Neumann wanted to keep his proof as general as possible, and therefore derived the statistical formula that we are about to discuss from his three assumptions and from the so called continuity assumption. This continuity assumption is a condition on states; it singles out so called *normal* states on the space of all bounded operators on our Hilbert space H. By *normal* in this context is meant 'σ-weak continuous' which has to do with (pointwise) convergence of quantum-mechanical expectation values. The σ-weak topology is provided by the seminorms $||A||_\rho = |\mathrm{Tr}(\rho A)|$, where ρ is an element of the space of all trace-class operators on H. Hence $A_n \to A$ σ-weakly when $|\mathrm{Tr}(\rho A_n)| \to |\mathrm{Tr}(\rho A)|$ for all trace-class ρ. At this point we could go on and on and write a whole book about all this, but we will just define what we mean by 'trace-class' operators and stop: for any bounded operator A in H, define the *trace norm* of A by $||A||_1 := \mathrm{tr}(|A|) = \mathrm{tr}(\sqrt{(}a^*a))$ (which is $= \sum_i \langle e_i, Ae_i \rangle$, where the e_i form an orthonormal basis of H). A trace-class operator is a bounded operator A from H, satisfying $||A||_1 < \infty$.

Using this continuity condition implicitly together with his three assumptions von Neumann arrived (from 'first principles' as one says) at his statistical formula (we already encountered it before)

$$\langle \mathcal{R} \rangle = \mathrm{Tr}(UR). \tag{2.8}$$

A dispersion-free ensemble would satisfy the equation $\langle R^2 \rangle - \langle R \rangle^2 = 0$ for all R which means that $\mathrm{Tr}(UR^2) = (\mathrm{Tr}(UR))^2$. For R we can choose whatever we want, so why not take the projection operator, $P_\phi = |\phi\rangle\langle\phi|$. In this way von Neumann obtained $\mathrm{Tr}(UP)$ = $(\mathrm{Tr}(UP))^2$. This implies $\langle \phi, U\phi \rangle = \langle \phi, U\phi \rangle^2$, for all ϕ. Therefore either $U = 0$ or $U = 1$. Von Neumann concludes that since both conclusions are unacceptable, the non-existence of dispersion-free ensembles has been demonstrated.

Von Neumann now turns to the question of existence of *homogeneous ensembles*, being ensembles E such that for every physical quantity R and every pair of subensembles E_1, E_2 the equation $\langle R \rangle_E = a\langle R \rangle_{E_1} + b\langle R \rangle_{E_2}$ with $a > 0$, $b > 0$, and $a + b = 1$ implies that $\langle R \rangle_E = \langle R \rangle_{E_1} = \langle R \rangle_{E_2}$. He showed that an ensemble E is homogeneous if and only if its statistical operator U is a projection operator $P_\psi = |\psi\rangle\langle\psi|$ (up to a constant factor), from which he demonstrates the existence of homogenous ensembles; see [35] for further details. Having done all this, von Neumann was in the position to think that he proved the completeness of quantum mechanics.

2.7.1 Bell's reaction

Bell did not agree on everything with von Neumann. This is his reasoning: Let A and B be self-adjoint operators. Then $C = \alpha A + \beta B$ (with $\alpha, \beta \in \mathbb{R}$) is also self-adjoint and for the expectation values we have

$$\langle C \rangle = \alpha\langle A \rangle + \beta\langle B \rangle. \tag{2.9}$$

Consider now a 'hidden state' V. We can derive from V 'expectation values' $\langle A \rangle_V$, which, in general, do not equal the quantum mechanical ones ($\langle A \rangle_V \neq \langle A \rangle$). If we now require, as von

Neumann does, that these ‘hidden state expectation values’ also conform to

$$\langle C\rangle_V = \alpha\langle A\rangle_V + \beta\langle B\rangle_V, \tag{2.10}$$

we get into trouble. Consider the following example, which shows that (2.10) cannot be true. Let $A = \sigma_x$ and $B = \sigma_y$. Then operator $C = (\sigma_x + \sigma_y)/\sqrt{2}$ corresponds to the observable of the spin component along the direction bisecting x and y. Now we know that (in suitable units) all spin components have possible values 1 and -1 only. Therefore the hidden variable proponent has to ascribe ± 1 to the ‘expectation values’ of A, B and C. This implies the equality

$$\pm 1 = (\pm 1 + \pm 1)/\sqrt{2},$$

which is false. So this example illustrates why von Neumann’s argument is unsatisfying. However, the step from (2.9) to (2.10) can be performed for compatible observables with von Neumann’s third assumption. (Remember that compatible observables are those observables which, according to quantum mechanics, are jointly measurable in one arrangement.)

Later, it became clear through Gleason’s work [19] that von Neumann’s result concerning the impossibility of hidden variables does not hinge on the third assumption. In Hilbert spaces of at least three dimensions it suffices to postulate such an additivity rule for commuting operators *alone* in order to exclude the possibility of dispersion-free states.

Jauch and Piron reformulated [26] von Neumann’s proof using so called “quantum logic”. Unfortunately, it would require too much time to discuss this proof, for the reader is not assumed to be familiar with lattice theory.

2.8 The Kochen-Specker theorem

In 1967 Kochen and Specker published a proof [27] on the impossibility of hidden variables, that was based on new arguments. The Kochen-Specker theorem establishes a contradiction between the combination of value definiteness (VD) + non-contextuality (NC) and quantum mechanics, where VD and NC are defined as follows:

1. (VD) All observables defined for a quantum system have definite (i.e., sharp) values at all times.
2. (NC) If a quantum system possesses a property (value of an observable), then it does so independently of how that value is eventually measured.

In short we can put it like this. Acceptance of quantum theory forces us to abandon either (VD) or (NC). The problem for hidden variable theories is that (VD) is the key motivation for such theories, whereas it seems impossible to come up with a quantum theory containing (VD), but not (NC).

Since the Kochen-Specker theorem is one of the major no-hidden-variable theorems of quantum mechanics, we will study it in some more detail. First some notation. We distinguish properties $A, B, C, \ldots$ from the operators that represent them, denoted by $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$. Further, let $v(A), v(B), v(C), \ldots (\in \mathbb{R})$ be the values of properties $A, B, C, \ldots$ in a given state. The Kochen-Specker theorem states that for Hilbert spaces $\mathcal{H}$ of dimension ≥ 3, the following are contradictory (the Value Constraints are (in part) a consequence of VD):

1. (VD) Any set of properties $A, B, C, \ldots$ represented by operators $\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$, on $\mathcal{H}$ simultaneously have values $v(A), v(B), v(C), \ldots$.
2. (Value Constraints)

   (a) If A, B, C are compatible properties and $C = A + B$, then $v(C) = v(A) + v(B)$. (Sum Rule)

   (b) If A, B, C are compatible properties and $C = AB$, then $v(C) = v(A)v(B)$. (Product Rule)

Note that 2(b) is a very strong assumption; in classical mechanics it is only true for pure states. Recall that compatibility means that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ all have a complete set of eigenvectors in common.

We can recast the KS theorem as the claim that (VD) and (PVC) are contradictory, where PVC stands for Projection operator Value Constraint, which means the following:

(PVC) Let $|a_1\rangle, |a_2\rangle, \ldots, |a_N\rangle$ form an orthonormal basis for an N-dimensional Hilbert space. Then, $v(P_{|a_1\rangle}) + v(P_{|a_2\rangle}) + \ldots + v(P_{|a_N\rangle}) = 1$, where $v(P_{|a_i\rangle}) = 1$ or 0, for $i = 1, \ldots, N$. (Here P is the projection operator defined by $P_{|\Psi\rangle}|\Phi\rangle \equiv \langle\Psi|\Phi\rangle|\Psi\rangle$.)

To see that PVC follows from the (Value of Constraints) part of the KS theorem we have to recall that

$$
\begin{aligned}
(P_{|\Psi\rangle})^2|\Phi\rangle &= P_{|\Psi\rangle}(P_{|\Psi\rangle}|\Phi\rangle) = P_{|\Psi\rangle}(\langle\Psi|\Phi\rangle|\Psi\rangle) = \langle\Psi|\langle\Psi|\Phi\rangle|\Psi\rangle|\Psi\rangle \\
&= \langle\Psi|\Phi\rangle\langle\Psi|\Psi\rangle|Psi\rangle = \langle\Psi|\Phi\rangle|\Psi\rangle = P_{|\Psi\rangle} \;\; \text{(idempotency)}
\end{aligned}
$$

and, from the fact that $\langle a_i|a_j\rangle = 0$, unless $i = j$, we have that

$$
P_{|a_1\rangle} + P_{|a_2\rangle} + \ldots + P_{|a_N\rangle} = I_N \;\; \text{(resolution of identity)},
$$

where $|a_1\rangle + |a_2\rangle + \ldots + |a_N\rangle$ forms an orthonormal basis for an N-dimensional Hilbert space $\mathcal{H}$ with identity operator I_N.

In fact (VD) requires that (PVC) holds not only for the $|a_i\rangle$, orthonormal basis, but for all orthonormal bases. This is because (PVC) says: The operator A with eigenvectors $|a_i\rangle$ has a definite value, and (VD) says: *All* operators have definite values (hence even those incompatible with A). The proof of the KS theorem is now that we cannot have (PVC) for all orthonormal bases of Hilbert spaces with dimension ≥ 3.

We now might implement (PVC) by the following: Label each $P_{|a_i\rangle}$ either black or white, depending on whether $v(P_{|a_i\rangle}) = 1$ or 0. Then from (PVC) for the set of bases corresponding to $P_{|a_i\rangle}$, one is labeled white and all the others are labeled black. But now (VD) requires us to do this for all sets of bases. What we need to demonstrate is that one cannot color all the bases of a Hilbert space of dimension ≥ 3 such that each one member of each set is white and all the other members of the set are black. We do not prove this here, but from the figure it should be clear that it does work for dimension 2, but does not work for dimension 3.

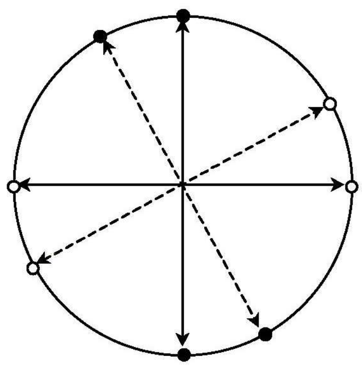

Figure 1: If we continue colouring as in the diagram, we will color the entire circle such that each alternating quadrant is black or white.

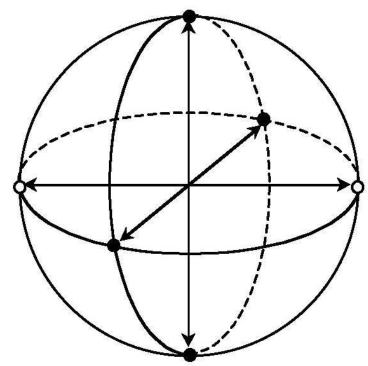

Figure 2: One cannot consistently color the entire surface of the sphere in this manner. At some point, you will run over a previously coloured portion!

After having looked at some of the important no-hidden-variables theories, we will now discuss 'the new era in no-hidden-variable- theories' initiated by John Bell.

3 Bell's inequality

The main subject of this chapter, Bell's inequality, was inspired by the EPR paradox. (We will see why we call it a paradox later on.) Einstein, Podolsky, and Rosen stated that quantum mechanics was incomplete. As mentioned in chapter 2, this paper inspired some physicist to find a completion of the theory. These theories became known as hidden variable theories. Such theories were meant to be compatible with EPR's locality and separability principles. Von Neumann though, tried to prove such completions could not exist, as we have seen in section 2.7. Bell made a huge contribution to the discussion, firstly by giving counterexamples of previous 'no go' theorems, and secondly by replacing all the foregoing theorems by a new one that is accepted among all physicists today. He published these results in an article [5] that was published in 1966. He took a close look at the assumptions of all foregoing 'no go' theorems and concluded that those theorems have very strong assumptions. Therefore they rule out only a very small class of hidden variable theories. Even before his comments on the previous 'no go' theorems came out (due to a long delayed publication), Bell had published his famous article [4] about what we now call Bell's theorem in 1964. There he proved that no hidden variable theory satisfying locality constraints is compatible with quantum mechanics. A lot of insights mentioned throughout this chapter are based on Bub's book [9].

3.1 Explaining correlations

Suppose we have two possible events a and b. For example, a is the event in which the spin of an electron is measured in the z-direction, and a value of 1 is found. Typically such events would have a certain probability of occurring, say $p(a)$ and $p(b)$. If the events happen to be totally unrelated, the probability $p(a \,\&\, b)$ of both events occurring would be obtained by multiplying the individual probabilities. If this is *not* the case, i.e.,

$$p(a \,\&\, b) \neq p(a)p(b), \tag{3.1}$$

then we call the events *correlated*. A correlation between two events a and b is called *perfect*, if b occurs whenever a does. In terms of probabilities: a correlation is perfect when $p(a \,\&\, \overline{b}) = 0$ and $p(\overline{a} \,\&\, b) = 0$, where $\overline{b}$ denotes the event that b does not occur.

Let us consider the version of the EPR problem with two spin-$\frac{1}{2}$ particles in the singlet state.[6] Quantum mechanics predicts that every time one measures the spin of both particles in the same direction, one is sure to find opposite values. But the individual probability of finding a certain value for either of the particles is only 50 percent. If the two events of finding a certain value for a particle were uncorrelated, one would expect to find the opposite value 50 percent of the time. Thus the events are correlated, and the correlation is perfect.

[6]This version of the EPR problem was introduced by Bohm ([7] pp. 614–619) to simplify the mathematics involved.

Such a correlation calls for an explanation. Van Fraassen offers [16] six possible ways of attempting this:

1. Chance
2. Coincidence
3. Pre-established harmony
4. Co-ordination
5. Logical identity
6. Common cause

If two events have a certain probability of occurring, there is always a chance that they will coincide. And if one observes them a finite number of times, there is always a chance that they will coincide every time. So if you perceive a certain correlation between events, it could be just because of a twist of fate. This is an explanation by means of *chance*. Such an explanation would imply that the correlation will not persist indefinitely. Of course such an explanation of the correlations in EPR would mean that quantum mechanics is wrong and that in fact there is no correlation. The chance that the correlation observed in EPR like situations is pure *chance*, is of course extremely small, because of the great number of observations.

Two events could also coincide for totally separate reasons. For example suppose you have a friend on whose birthday fireworks are always lit, but the fireworks are not meant to celebrate his birthday. In fact, the fireworks have nothing to do with his birthday, and if his birthday had not been the first of January, there probably never would be fireworks on his birthday. Such an explanation is called *coincidence*. If an observed correlation is coincidental, it could persist or it could not (maybe fireworks will be forbidden at some point in time). Such an explanation is not possible for the correlations observed in quantum mechanics, since we expected the correlation to hold for every pair of electrons created in the singlet state.

The *pre-established harmony* explanations cover what is also sometimes called “God did it” type arguments. This refers to the way Young Earth Creationists respond to the presence of dinosaur bones in the earth's crust, which appear to be much and much older than the age of the earth according to the Bible. They just say that God put them there when he created the world. The same sort of scheme could be used to explain the EPR correlations. There could be some divine entity who controls what happens, and has decided that these events had to be correlated in the way they are. No further explanation is needed, because God needs no reasons.

There is no way to dismiss such an argument, which of course could be less religiously motivated. The more physical counterpart of this sort of explanation usually requires the universe to have started in some specific state, which explains how the correlations came to be through the natural evolution of the universe. This sort of argument is almost as shunned by physicists as the religious one. [7]

Leaving the realm of pure chance and divine influence we now come to the more physical explanations of correlations. The first is *co-ordination*. It could be that some sort of signal is sent when the first event occurs, which in some way affects the probability of the second event happening. This sort of explanation runs into trouble when the events take place at points in space-time with space-like separation. In such a situation there is no way to tell which of the events came first. So it is unclear which event is supposed to affect the outcome of the other. In spite of these objections, this might turn out to be the correct explanation, as will be discussed in chapter 4.

An explanation by *logical identity* argues that two variables are correlated because they are (aspects of) the same thing. Suppose somebody drives a car with two speed indicators, one in miles per hour and one in kilometres per hour. He will find a correlation between the two speeds. One is always 1.6 times the other. The reason for this correlation is quite clear: both indicators measure the same thing, the speed of the car.

In more general mathematical terms, an explanation by logical identity states that variables A and B are correlated because both may be defined as functions f and g of a third variable C.

Of course, the relations of A and B to C are, a priori, not any less mysterious than the relation between A and B. In the case of correlated variables in space-like separated systems logical identity would give no real explanation if the variable C belonged to a system that had a space-like separation to the system of either A or B (in this case we would still have a correlation between variables in space-like separated systems). So the only real possibilities here are that either C belongs to a system in the common past of the systems of A and B. In which case it could be considered a common cause, which we will treat later on. Or, we must accept that there variables which may be observed simultaneously in space-like separated systems, which is the option that is usually meant when logical identity is referred to.

Van Fraassen argues that in fact it was this sort of explanation that was explored in some early reactions to the EPR paper. And he adds that they are excluded by 'no go' results of von Neumann, Jauch, Gleason, Kochen and Specker, and Bell. We will not go into this any further here. For a more detailed account see [16], mainly chapter 10.

Now we come to the last of the six explanations: *common cause*. It could be that both events are caused by an event in their common history. For example the chance of my house burning down is (luckily) quite small. The same is true for my neighbour's house. But if my house burns down, there is a quite significant chance that my neighbour's house will also burn down. So the events of my house burning down and my neighbour's house burning down are correlated. The

[7] Although shunned by physicists such an explanation cannot be disregarded a priori. But we will see that this last sort of explanation amounts to the sort of hidden variable theory that is discounted by the work of Bell. (The starting state of the universe would function as a hidden variable.)

correlation is quite easily explained by the fact, that if there is a fire on my block, there is a good chance of it affecting both houses. The events of the two houses burning down are both the result of the same cause namely the fire on my block.

It is this sort of explanation (by common cause), that is addressed by Bell's inequality, and that we will explore in the rest of this chapter.

3.2 Making sense of causality in an indeterministic world

In a deterministic world it is quite clear what we mean by causality. A complete causal account of the world would be one in which every event may be explained as the result of previous events. But this notion is useless in an indeterministic world, where by its very nature events may happen spontaneously for no apparent reason. Reichenbach provided a new criterion: there should be a causal explanation not of every individual event, but for every correlation.

In this perspective a *common cause* for the events a and b is an event c that satisfies the following conditions:

1. c precedes a and b.
2. $p(a|c) > p(a|\bar{c})$ and $p(b|c) > p(b|\bar{c})$.
3. $p(a|b \,\&\, c) = p(a|c)$ and $p(b|a \,\&\, c) = p(b|c)$.

Condition 3 expresses that c not only raises the probabilities of a and b, but also restores their statistical independence. Once the common cause c is specified, the probabilities for a and b (conditional on c) should be independent of each other. This will turn out to be a crucial property for the proof of Bell's inequality.

3.3 Common causes and local hidden variables

We will now see how the concept of a common cause fits into the picture of the EPR correlations. For now, we assume the locality principle of EPR (see 2.6). Suppose we have two spatially separated systems, labelled L(eft) and R(ight) (e.g. two spin-$\frac{1}{2}$ particles moving in opposite directions). Both systems have a number of observables with two possible values (e.g. spin in a certain direction). The event that a certain observable A is measured in L will be denoted by LA and so forth. Each measurement of an observable has two possible outcomes (0 and 1). The event that observable B is measured in R with outcome b will be denoted Rb and so on.

Suppose there is some correlation between the outcomes of measurements in L and R,

$$p(La \,\&\, Rb|LA \,\&\, RB) \neq p(La|LA \,\&\, RB)p(Rb|LA \,\&\, RB)^8. \tag{3.2}$$

[8] The notation here is the usual one for conditional probabilities.

suppose now, that there is some common cause event C which can be characterized by some parameter λ. The common cause event C characterized by λ will be denoted $C\lambda$. Since λ does not appear in the formulation of the problem and is therefore unknown to us, λ may be called a *hidden variable*.

The conditional probabilities should satisfy the following conditions:

1. Outcome Independence (OI):
   $p(La \,\&\, Rb|LA \,\&\, RB \,\&\, C\lambda) = p(La|LA \,\&\, RB \,\&\, C\lambda)\, p(Rb|LA \,\&\, RB \,\&\, C\lambda)$.

2. Parameter Independence (PI):
   $p(La|LA \,\&\, RB \,\&\, C\lambda) = p(La|LA \,\&\, C\lambda)$ and
   $p(Rb|LA \,\&\, RB \,\&\, C\lambda) = p(Rb|RB \,\&\, C\lambda)$.

3. Hidden Autonomy:
   $p(C\lambda|LA \,\&\, RB) = p(C\lambda)$.

Outcome independence — sometimes also referred to as causality or completeness — expresses the final condition needed for a common cause as defined above. Because L and R are space-like separated, what we do in L should not affect what happens in R, and vice versa. Specifically, our choice of measurement in L may not affect the outcome of the measurement in R. This notion of locality is reflected by the second condition, parameter independence. The third condition expresses the fact that the common cause precedes the measurements in L and R and therefore should not be affected by the choice of measurements.

3.3.1 Perfect correlations and determinism

Note that if we assume the correlation to be perfect, the conditions above require the relation between the common cause and the measurement outcomes to be deterministic. Namely suppose the outcomes of A and B are correlated in such a way that they always have the opposite result:

$$\begin{aligned} 0 &= p(L0 \,\&\, R0|LA \,\&\, RB) \\ &= p(L0 \,\&\, R0|LA \,\&\, RB \,\&\, C\lambda) \\ &= p(L0|LA \,\&\, RB \,\&\, C\lambda)\, p(R0|LA \,\&\, RB \,\&\, C\lambda) \\ &= p(L0|LA \,\&\, C\lambda)\, p(R0|RB \,\&\, C\lambda). \end{aligned} \tag{3.3}$$

So we have that

$$p(L0|LA \,\&\, C\lambda) = 0 \text{ or } p(R0|RB \,\&\, C\lambda) = 0. \tag{3.4}$$

And, in the same way,

$$p(L1|LA \,\&\, C\lambda) = 0 \text{ or } p(R1|RB \,\&\, C\lambda) = 0. \tag{3.5}$$

We also have that

$$p(R0|RB \,\&\, C\lambda) + p(R1|RB \,\&\, C\lambda) = 1, \text{ and}$$
$$p(L0|RB \,\&\, C\lambda) + p(L1|RB \,\&\, C\lambda) = 1. \tag{3.6}$$

The only way all of these statements can be true is if all probabilities are either one or zero. Thus in the case of a perfect correlation the common cause fully determines the outcome of any measurement of A and B, a fact that has sometimes been overlooked in discussions about the EPR correlations.

3.4 Bell's inequality

Bell showed with his inequality that no local hidden variable theory is compatible with quantum mechanics. Before we actually state the inequality and prove it, we present an example which serves to develop some intuition in the subject.

3.4.1 Example

As we know, when light passes two polarized filters, the amount of light that passes through the second filter is related to the polarization of the beam in between the filters:

$$\frac{E_1}{E_2} = \cos^2(\alpha), \tag{3.7}$$

where α is the angle of misalignment of the two filters. Two following cases will be of particular interest to us. When $\alpha = 30$ degrees, 75% of the beam gets through. When $\alpha = 60$ degrees, 25% of the beam gets through. This experiment can be explained by considering light as a wave. (By taking the parallel component of the polarization vector every time the light passes a filter.) But, as Einstein observed in 1905, light does not always behave like a wave. For example, light quanta can be counted by a photomultiplier tube. In this case we interpret formula (3.7) as the probability that a photon will pass the second filter, when it already has passed the first. In fact, which model we choose to describe light and whether it is correct doesn't matter, all we are interested in are the experimental results expressed in formula (3.7). We now describe an experiment that contradicts our intuitive picture of reality.

When excited electrons in a calcium atom cascade down to their ground state, they emit light. In particular, a pair of photons is sent in opposite directions. When a filter is placed in the trajectory of one of these photons, it will pass 50 percent of the times. The same holds for the other photon. But if one first looks at the first photon and sees that it passes the filter, the other one behaves exactly like a photon that has gone through a filter which was set as the filter through which the first photon went. In case of absorption of the first photon, the other acts as a photon with polarization of $\theta + 90^\circ$. Here θ is the angle of the first filter (with respect to some given angle).

Now we will look at a simpler experiment which will be sufficient for our purposes. We take the same experimental setting as above, but only allow ourselves to measure at three angles namely 0, 30 and 60 degrees. The behaviour of the photons can be summarized as follows:

1. When measured at the same angle the photons always display the same behaviour. (So they either both pass or are both absorbed.)

2. When measured at angles that differ by 30 degrees, the photons agree 25 percent of the time

3. When measured at angles that differ by 60 degrees, the photons agree 75 percent of the time.

The question is: can the photons make any agreement in order to reproduce these results, assuming they are not able to communicate after they have left each other. (This represents the locality and separability constraints.) First of all, they have to 'know' how to respond to each measurement, because they have to be sure that the other photon does exactly the same when the two measurements are equal. The second point is that they can gain no advantage by using some sort of random element once they have left each other, since they could just as well have 'flipped that coin' while still together and share the results with each other. Or flip the coin three times, one for each measurement. Therefore any strategy that involves local stochastic elements can do no better than a corresponding strategy in which the random choices are made at the source. So when they leave the calcium atom, each photon should know exactly what its response will be to every measurement. These two conditions cut our possibilities to the number of eight. Because both agree when measured at the same angle, we only need to say how the first photon responds to each measurement. There are eight possibilities, of which four are stated below. The other four are the mirror images of these outcomes. (e.g. (P, A, A)[9] is the mirror image of (A, P, P)):

1. (P, P, P)

2. (A, P, P)

3. (P, A, P)

4. (P, P, A)

Indeed, strategy 1 entails that the photon will pass the filter no matter which angle it is set on. Number two entails that the photons will be absorbed by a filter at 0 degrees and pass the other two. We stated four strategies above, but their mirror images are strategies as well. We are, however, only interested if the answers of the two photons agree or differ. Therefore (P, P, P)

[9] P stands for pass and A stands for absorb.

is the same as (A, A, A) for this purpose. In both cases they agree on all measurements. Now we shall see that there is no way of reproducing the results as described above (point 1, 2 and 3). Suppose the photons choose strategy 1 a percent of the time, strategy 2 b percent of the time, 3 c percent of the time, and 4 d percent of the time. Of course, $a + b + c + d = 100$, and all a, b, c, d are positive real numbers. We assume that the choice of strategy is completely independent of the choice of questions (hidden autonomy). In 75% of the time the photons have to disagree when one is measured at 0 degrees and the other at 60 degrees. So they have to choose strategy 2 or 4 in exactly 75% of the time. So $b + d = 75$. With similar arguments one can conclude:

$$c + d = 25 \tag{3.8}$$

$$b + c = 25 \tag{3.9}$$

$$b + d = 75 \tag{3.10}$$

Solving gives:

$$a = 37.5 \tag{3.11}$$

$$b = 37.5 \tag{3.12}$$

$$c = -12.5 \tag{3.13}$$

$$d = 37.5 \tag{3.14}$$

So c is negative, which can't possibly be true. So there is no long term strategy that the photon can discuss beforehand, so that their behaviour satisfies equation (3.7).

3.4.2 Proof of Bell's inequality

We have seen one particular case in which hidden variables cannot explain the observed behaviour. Bell's inequality gives a general condition, which should hold for any local deterministic hidden variable theory. First consider a pair of spin-$\frac{1}{2}$ particles. We can define two functions $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$. These functions give the results of respectively the spin measurement on particle 1 in direction **a** and on particle 2 in direction **b**. Note that these functions also depend on the hidden variable λ. We have:

$$A(\mathbf{a}, \lambda) = \pm 1 \quad ; \quad B(\mathbf{b}, \lambda) = \pm 1. \tag{3.15}$$

We now look at the average value of the product of the two components $A(\mathbf{a}, \lambda)$ and $B(\mathbf{b}, \lambda)$. We call this average $q(\mathbf{a}, \mathbf{b})$. If $\rho(\lambda)$ is the probability distribution of λ, this average value is

$$q(\mathbf{a}, \mathbf{b}) = \int \rho(\lambda) A(\mathbf{a}, \lambda) B(\mathbf{b}, \lambda) d\lambda. \tag{3.16}$$

We will now derive Bell's inequality. The arguments are based on Griffith's notes on this topic (see [21]). Since $q(\mathbf{d},\mathbf{d}) = -1$ for all **d** (when the detectors are aligned, the results are perfectly anti-correlated)

$$A(\mathbf{d},\lambda) = -B(\mathbf{d},\lambda) \text{ for all } \lambda. \tag{3.17}$$

Now we can rewrite our expression for $q(\mathbf{a},\mathbf{b})$ as

$$q(\mathbf{a},\mathbf{b}) = -\int \rho(\lambda)A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)d\lambda. \tag{3.18}$$

Let **c** be another unit vector. Then

$$q(\mathbf{a},\mathbf{b}) - q(\mathbf{a},\mathbf{c}) = -\int \rho(\lambda)[A(\mathbf{a},\lambda)A(\mathbf{b},\lambda) - A(\mathbf{a},\lambda)A(\mathbf{c},\lambda)]d\lambda, \tag{3.19}$$

and because $(A(\mathbf{b},\lambda))^2 = 1$,

$$q(\mathbf{a},\mathbf{b}) - q(\mathbf{a},\mathbf{c}) = -\int \rho(\lambda)[1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)]A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)d\lambda. \tag{3.20}$$

¿From equation (3.15) we conclude $-1 \leq [A(\mathbf{a},\lambda)A(\mathbf{b},\lambda)] \leq 1$ and
$\rho(\lambda)[1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)] \geq 0$, so:

$$|q(\mathbf{a},\mathbf{b}) - q(\mathbf{a},\mathbf{c})| \leq \int \rho(\lambda)[1 - A(\mathbf{b},\lambda)A(\mathbf{c},\lambda)]d\lambda, \tag{3.21}$$

or, equivalently,

$$|q(\mathbf{a},\mathbf{b}) - q(\mathbf{a},\mathbf{c})| \leq 1 + q(\mathbf{b},\mathbf{c}), \tag{3.22}$$

which is the famous Bell inequality. The only assumption is that our hidden variable theory is subject to equations (3.15) and (3.17). In other words, the assumptions are that we consider a pair of entangled particles and that the measurement results are 2-valued and predetermined by λ, a local hidden variable.

It is easy to see that quantum mechanics is incompatible with this inequality. Quantum mechanics predicts namely the following equation:

$$q(\mathbf{a},\mathbf{b}) = -\mathbf{a}\cdot\mathbf{b}. \tag{3.23}$$

For example, vectors that do not satisfy the inequality are the following: **a** and **b** are orthogonal and **c** makes an angle of 45 degrees with both **a** and **b**. In this case:

$$q(\mathbf{a},\mathbf{b}) = 0, \quad q(\mathbf{a},\mathbf{c}) = q(\mathbf{b},\mathbf{c}) = -0.707. \tag{3.24}$$

These results are clearly inconsistent with Bell's inequality:

$$0.707 \not\leq 1 - 0.707 = 0.293. \tag{3.25}$$

This sheds new light on the EPR paradox. If quantum mechanics is right, no hidden variable theory can explain the non-local character of quantum mechanics. At this point we only have to carry out a 'simple' experiment, in which systems of two spin-$\frac{1}{2}$ particles are repeatedly measured at different angles. Such experiments have been done and will be discussed in chapter 5.

3.5 The Greenberger-Zeilinger-Horne (GHZ) theorem

As we have seen, it is impossible for hidden variables to explain the results of certain spin measurements. In our last example we considered two photons, travelling in opposite directions. In this case the outcome of a single measurement is not enough to ascertain that a local theory does not suffice. We have to observe their behaviour in the long series of experiments.

In 1989 Greenberger, Zeilinger and Horne [20] found an example of a quantum mechanical state in which no local theory can explain the behaviour in a single experiment.

GHZ looked at three spin-$\frac{1}{2}$ particles. Just as in two particle case, the three particles start together and then move away from each other. After a while, either their x-spin or y-spin is measured. GHZ found there is a (3-particle) state $|\varphi\rangle$ that has the following properties: if any of the four observables on the left of table 1 is measured, the outcome will be the number on the right with certainty. (So $|\varphi\rangle$ is an eigenstate of these four observables.) We denote the outcome of a measurement of the x-spin of particle 2 as X_2.

Observable	Value
$X_1Y_2Y_3$	-1
$Y_1X_2Y_3$	-1
$Y_1Y_2X_3$	-1
$X_1X_2X_3$	1

Table 1

We will clarify what we exactly mean by this in due course. But the most interesting feature of this table is that there is no strategy the particles can agree on, to reproduce this result, because before the particles leave each other they already have to know their answer to each question. And it is impossible to solve the four equations in table 1. This can easily be seen by multiplying all equations. The left hand side will be 1, because all the outcomes of spin measurements will be ± 1 and every term appears twice on the left. The right hand side equals -1.

We consider a composite three spin-$\frac{1}{2}$ particle (S_1, S_2, S_3) system. Our Hilbert space is therefore $\mathcal{H}(S_1) \otimes \mathcal{H}(S_2) \otimes \mathcal{H}(S_3)$. Next we consider the following observables:

$$\begin{gathered}
\sigma_x^1 \otimes I_2 \otimes I_3, \sigma_y^1 \otimes I_2 \otimes I_3; \\
I_1 \otimes \sigma_y^2 \otimes I_3, I_1 \otimes \sigma_x^2 \otimes I_3; \\
I_1 \otimes I_2 \otimes \sigma_x^3, I_1 \otimes I_2 \otimes \sigma_y^3; \\
\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3; \\
\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3; \\
\sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3; \\
\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3.
\end{gathered} \tag{3.26}$$

The last four operators can all be written as the product of three of the six other operators and those three operators commute. For example:

$$\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 = \sigma_y^1 \otimes I_2 \otimes I_3 \cdot I_1 \otimes \sigma_x^2 \otimes I_3 \cdot I_1 \otimes I_2 \otimes \sigma_y^3. \quad (3.27)$$

It is clear that the three operators on the right-hand side commute. It is also easily seen that they all commute with the operator on the left side. In quantum mechanics one accepts that if commuting operators satisfy a relation as in (3.27), their eigenvalues satisfy the same relation. When we denote the eigenvalue of operator A in state φ by $ev_\varphi(A)$, we obtain:

$$ev_\varphi(\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3) = ev_\varphi(\sigma_y^1 \otimes I_2 \otimes I_3) \cdot ev_\varphi(I_1 \otimes \sigma_x^2 \otimes I_3) \cdot ev_\varphi(I_1 \otimes I_2 \otimes \sigma_y^3). \quad (3.28)$$

And because the eigenvalues of the six first operators of list (3.26) are clearly ± 1 so are the eigenvalues of the last four.

The next objective is to show that the last four operators of list (3.26) commute. Therefor we need the following relations for spin operators:

$$(\sigma_x)^2 = (\sigma_y)^2 = (\sigma_z)^2 = i, \quad (3.29)$$
$$\sigma_x \sigma_y = i\sigma_z, \quad (3.30)$$
$$\sigma_y \sigma_x = -i\sigma_z. \quad (3.31)$$

We can now show that $\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3$ and $\sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3$ commute:

$$\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \cdot \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 = I_1 \otimes \sigma_z^2 \otimes \sigma_z^3 = \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 \cdot \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \quad (3.32)$$

By similar calculations all bottom four operators of list (3.26) commute. Therefore they have a common set of eigenvectors. Let $|\psi\rangle$ be such an eigenvector. At this point Einstein would have argued that the measurement of a spin-component of a particle cannot depend on the other two. So the outcome of the measurement has to be fixed already. We recall our notation from the beginning of this section. Y_3 is the outcome of the measurement of the y-spin of particle 3. Now the equations in table 1 have to hold all four and this is impossible as we have seen.

In conclusion, GHZ assume (1) locality and separability (2) the eigenstate-eigenvalue relation and (3) relations between commuting operators hold for their respective eigenvalues. They conclude (1) is not consistent with (2) and (3). This argument shows that either the orthodox interpretation of quantum mechanics or the locality and separability principle of EPR is false. And again we can 'easily' check that by carrying out the experiment. But in this case the experiment is so difficult to perform that nobody has yet carried it out successfully.

Although the elegance of doing a single experiment to settle the EPR discussion is tempting, the results of it would not be as strong as those obtained by testing Bell's inequality, because the assumptions made by GHZ are quite strong. Especially assumption (3) has been the target of much critique, since it is not at all clear that it should hold for any hidden variable theory.

3.6 The Clauser-Horne (CH) inequalities

In this section we will derive a slightly more general version of Bell's inequality known as the Clauser-Horne inequalities. These were first derived in 1974 by Clauser and Horne [11] to arrive at an inequality that would be more suitable for experimental testing. The derivation as presented here is based on the one given by Bub in [9].

In this section we will use a more compact notation then before. We again have two spatially separated systems L and R. Each system again has a number of observables denoted by A, A', A'',... and B, B', B'',... respectively. Each of the observables is supposed to possess two possible values, a_+ and a_- for A, a'_+ and a'_- for A', and similarly for the other observables. The probability that a value a is detected for A, while measuring A in L and B in R is denoted by $p^{AB}(a)$ (here a is the variable which could be either a_+ or a_-). The joint probability for finding the value a for A and b for B in the same situation will be denoted by $p^{AB}(a \,\&\, b)$.

Note that in the case of $p^{AB}(a \,\&\, b)$ the superscript indicating the observable is redundant, because this information is already contained in the results. In some cases we will drop the redundant superscripts to allow for an even more compact notation.

The systems L and R are again supposed to be in some entangled state which causes the measurement outcomes in L and R to be correlated. So:

$$p^{AB}(a \,\&\, b) \neq p^{A}(a)p^{B}(b). \tag{3.33}$$

Suppose now that this correlation can be explained by a common cause, which may be characterized by some parameter λ. The probabilities conditional on λ having a certain value will be denoted by a λ in the subscript. Compared to the notation in 3.3 we now have for example:

$$p_\lambda^{AB}(a \,\&\, b) = p(La \,\&\, Rb | LA \,\&\, RB \,\&\, C\lambda)$$

For some suitable measure μ representing the distribution of λ with the property that

$$\int d\mu(\lambda) = 1, \tag{3.34}$$

the total probabilities for a certain outcome may be expressed by

$$p^{AB}(a) = \int p_\lambda^{AB}(a) d\mu(\lambda). \tag{3.35}$$

Translated into the new notation the conditions of parameter independence and outcome independence become respectively:

$$\begin{aligned} p_\lambda^{AB}(a) &= p_\lambda^{A}(a), \\ p_\lambda^{AB}(b) &= p_\lambda^{B}(b), \end{aligned} \tag{PI}$$

and

$$p_\lambda^{AB}(a \,\&\, b) = p_\lambda^{AB}(a)p_\lambda^{AB}(b). \tag{OI}$$

which by definition of conditional probability is equivalent to the following two conditions:

$$p_\lambda^{AB}(a|b) = p_\lambda^{AB}(a), \tag{OIa}$$
$$p_\lambda^{AB}(b|a) = p_\lambda^{AB}(b). \tag{OIb}$$

Here $p_\lambda^{AB}(a|b)$ represents the conditional probability of detecting the value a for A when b has already been detected for B.

Taken together (PI) and (OI) are equivalent to the following condition:

$$p_\lambda^{AB}(a \,\&\, b) = p_\lambda^{A}(a) p_\lambda^{B}(b), \tag{SI}$$

which is known as conditional statistical independence (SI).

That (SI) is implied by (PI) and (OI) is easily seen by applying (PI) to (OI). The other way around is a little less trivial. First we will show that (SI) implies (PI). (SI) holds specifically for b_+ and b_-:

$$\begin{aligned} p_\lambda^{AB}(a \,\&\, b_+) &= p_\lambda^{A}(a) p_\lambda^{B}(b_+) \\ p_\lambda^{AB}(a \,\&\, b_-) &= p_\lambda^{A}(a) p_\lambda^{B}(b_-) \end{aligned} \tag{3.36}$$

Adding both equations yields:

$$\begin{aligned} p_\lambda^{AB}(a) &\equiv p_\lambda^{AB}(a \,\&\, b_+) + p_\lambda^{AB}(a \,\&\, b_-) \\ &= p_\lambda^{A}(a)[p_\lambda^{B}(b_+) + p_\lambda^{B}(b_-)] \\ &= p_\lambda^{A}(a). \end{aligned} \tag{3.37}$$

A similar derivation yields the other half of (PI). We now also see that (OI) follows if we apply (PI) to (SI).

Using (SI) we will now derive an inequality. Consider the expression

$$\begin{aligned} K &= \tfrac{\alpha}{A}[\alpha'(B-\beta) + (A-\alpha')(B-\beta')] + \tfrac{(A-\alpha)}{A}[\alpha'\beta' + (A-\alpha')\beta] \\ &= \alpha B + A\beta + \alpha'\beta' - \alpha\beta - \alpha'\beta - \alpha\beta'. \end{aligned} \tag{3.38}$$

If $0 \leq \alpha, \alpha' \leq A$ and $0 \leq \beta, \beta' \leq B$, then because $\alpha' + (A - \alpha') = A$ we have:

$$\begin{array}{lcl} 0 \leq & \alpha'(B-\beta) + (A-\alpha')(B-\beta') & \leq AB, \\ 0 \leq & \alpha'\beta' + (A-\alpha')\beta & \leq AB. \end{array}$$

And from this and $\frac{\alpha + (A-\alpha)}{A} = 1$ we see:

$$0 \leq K \leq AB. \tag{3.39}$$

Let:

$$\begin{aligned} \alpha &= p_\lambda(a), & \beta &= p_\lambda(b), \\ \alpha' &= p_\lambda(a'), & \beta' &= p_\lambda(b'), \\ A &= 1 & B &= 1. \end{aligned}$$

where we have suppressed the redundant superscripts. Then:

$$0 \leq p_\lambda(a) + p_\lambda(b) + p_\lambda(a')p_\lambda(b') - p_\lambda(a)p_\lambda(b) - p_\lambda(a')p_\lambda(b) - p_\lambda(a)p_\lambda(b') \leq 1. \quad (3.40)$$

If we now assume conditional statistical independence (SI), we obtain

$$0 \leq p_\lambda(a) + p_\lambda(b) + p_\lambda(a' \,\&\, b') - p_\lambda(a \,\&\, b) - p_\lambda(a' \,\&\, b) - p_\lambda(a \,\&\, b') \leq 1. \quad (3.41)$$

Now applying (3.34) and (3.35) we find

$$0 \leq p(a) + p(b) + p(a' \,\&\, b') - p(a \,\&\, b) - p(a' \,\&\, b) - p(a \,\&\, b') \leq 1. \quad (3.42)$$

Exchanging a and a' and/or b and b', will yield similar inequalities for a total of four:

$$\begin{aligned} 0 \leq \quad & p(a) + p(b) + p(a' \,\&\, b') - p(a \,\&\, b) - p(a' \,\&\, b) - p(a \,\&\, b') & \leq 1, \quad (3.43a) \\ 0 \leq \quad & p(a') + p(b) + p(a \,\&\, b') - p(a' \,\&\, b) - p(a \,\&\, b) - p(a' \,\&\, b') & \leq 1, \quad (3.43b) \\ 0 \leq \quad & p(a) + p(b') + p(a' \,\&\, b) - p(a \,\&\, b') - p(a' \,\&\, b') - p(a \,\&\, b) & \leq 1, \quad (3.43c) \\ 0 \leq \quad & p(a') + p(b') + p(a \,\&\, b) - p(a' \,\&\, b') - p(a \,\&\, b') - p(a' \,\&\, b) & \leq 1. \quad (3.43d) \end{aligned}$$

These are known as the Clauser-Horne inequalities.

To complete this as a 'no go' theorem for hidden variables, we must find at least one situation in which quantum mechanics violates at least one of the above inequalities. Preferably this situation should be such that it may be tested experimentally.

Consider two spin-$\frac{1}{2}$ particles (L and R) created in the singlet state $\frac{1}{2}\sqrt{2}(\uparrow\downarrow - \downarrow\uparrow)$ moving in opposite directions. Let A and A' be the spin in z and x directions of the first particle, and let B and B' represent the spin of the other particle in two orthogonal directions in the zx-plane so that the angle between the direction of B and the z-axis is θ.

We now have a situation like the one used in the derivation of the Clauser-Horne inequalities. If $\theta = 0$, then

$$p(a \,\&\, b) = 0 \neq \frac{1}{2} \cdot \frac{1}{2} = p(a)p(b). \quad (3.44)$$

We will show that for certain angles θ at least one of the Clauser-Horne inequalities is violated by the predictions of quantum mechanics.

Let $|S,\phi\rangle$ denote the positive spin state in the ϕ direction in the system S, where S can be either L or R. In this notation the singlet state becomes:

$$\Phi = \frac{1}{\sqrt{2}}(|L,0\rangle|R,\pi\rangle - |L,\pi\rangle|R,0\rangle). \tag{3.45}$$

The probability of finding a positive value when measuring the spin in the ϕ direction is

$$\begin{aligned} ||S,\phi\rangle\langle S,\phi|\Phi|^2 &= \frac{1}{2}(|\langle S,\phi|S,0\rangle|^2 + |\langle S,\phi|S,\pi\rangle|^2) \\ &= \frac{1}{2}(|\langle S,0|S,\phi\rangle|^2 + |\langle S,\pi|S,\phi\rangle|^2). \end{aligned} \tag{3.46}$$

This last expression is just a half times the probability of finding either spin up or spin down, when the system is in the positive spin state in the ϕ direction. This probability of course is 1, because these are all the possible outcomes of the spin measurement. This tells us that

$$p(a) = p(a') = p(b) = p(b') = \frac{1}{2}. \tag{3.47}$$

The probability of finding two positive results, when measuring spin in the ϕ_L direction in L and spin in the ϕ_R direction in R, is the probability of getting a positive result for the measurement in L times the probability of finding a positive value for the the spin in the ϕ_R direction in R, when we know that L is in the positive ϕ_L state. The first probability is again $\frac{1}{2}$. For the second probability we know that is L is in the positive ϕ_L state, so R must be in the negative ϕ_L or $\phi_L + \pi$ state. So the total probability is:

$$\frac{1}{2}\left|\langle\phi_R|\phi_L+\pi\rangle\right|^2 = \frac{1}{2}\left|\langle -\pi|\phi_L-\phi_R\rangle\right|^2, \tag{3.48}$$

where $|\phi\rangle$ denotes the positive spin state in ϕ direction. Now let $\phi' = \phi_L - \phi_R$. In the standard Pauli matrix representation of spin $|-\pi\rangle$ is $\binom{0}{1}$. $|\phi'\rangle$ is the eigenvector with eigenvalue 1 of the spin operator in the ϕ' direction:

$$e_{\phi'}\cdot\vec{S} = \cos(\phi')\sigma_z + \sin(\phi')\sigma_x = \begin{pmatrix}\cos(\phi') & \sin(\phi') \\ \sin(\phi') & -\cos(\phi')\end{pmatrix} \tag{3.49}$$

thus:

$$|\phi'\rangle = \frac{1}{\sqrt{2-2\cos(\phi')}}\begin{pmatrix}\sin(\phi') \\ 1-\cos(\phi')\end{pmatrix} \tag{3.50}$$

and

$$\begin{aligned} \frac{1}{2}\left|\langle -\pi|\phi'\rangle\right|^2 &= \frac{1}{2}\left|\begin{pmatrix}0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2-2\cos(\phi')}}\begin{pmatrix}\sin(\phi') \\ 1-\cos(\phi')\end{pmatrix}\right|^2 \\ &= \frac{1}{2}(\frac{1}{2}-\frac{1}{2}\cos(\phi')). \end{aligned} \tag{3.51}$$

This gives us the joint probabilities:

$$\begin{aligned}
p(a \,\&\, b) &= \frac{1}{4} - \frac{1}{4}\cos(\theta) \\
p(a' \,\&\, b') &= \frac{1}{4} - \frac{1}{4}\cos(\theta) \\
p(a' \,\&\, b) &= \frac{1}{4} - \frac{1}{4}\cos(\theta - \frac{\pi}{2}) \\
&= \frac{1}{4} - \frac{1}{4}\sin(\theta) \\
p(a \,\&\, b') &= \frac{1}{4} - \frac{1}{4}\cos(\theta + \frac{\pi}{2}) \\
&= \frac{1}{4} + \frac{1}{4}\sin(\theta)
\end{aligned} \tag{3.52}$$

.

The second Clauser-Horne inequality (3.43b) becomes:

$$0 \le \frac{1}{2}(1 + \sin(\theta) + \cos(\theta)) \le 1, \tag{3.53}$$

which is false for $0 < \theta < \frac{\pi}{2}$.

The strength of the Clauser-Horne inequalities is that their derivation assumes very little about the nature of the correlations or the theory that produced them, like with Bell. The assumptions are just that there are two 2-valued observables for which the probabilities show a correlation, and that there exists a hidden variable account for this correlation that satisfies the (OI) and (PI) conditions. That is to say, the correlation is caused by some common cause.

The Clauser-Horne inequalities also assume very little about the nature of the common cause, besides it being local. In particular, they allow for indeterministic common causes. Of course, this is of little significance if one accepts quantum mechanics, because quantum mechanics predicts perfect correlations, and, as we have seen in section 3.3.1, perfect correlations require deterministic common cause explanations. But we can never empirically prove that the correlations are perfect, so it is good to leave this option open.

3.7 Conclusion

The EPR argument forces us to choose between causality and locality on one side and the completeness of quantum mechanics on the other. For a long time this issue has been the object of philosophical metaphysical debate, since there was no objective (empirical) way to choose between the two. The great achievement of Bell was that he gave an empirically verifiable criterion that distinguishes between the two options.

In this chapter we reviewed Bell's original result as well as two other results, the GHZ argument and the CH inequalities, that serve the same purpose. One of the elegant points of the argument used by Bell to obtain his inequality is that he assumes very little about the nature of quantum mechanics or any underlying hidden variable theory. The assumptions about the hidden variable theory are just the one we want to test, namely locality and causality. The only possible point of critique is that Bell does assume one thing about quantum mechanics: namely that the correlations encountered are perfect.

It is this point where the Clauser-Horne inequalities form a real improvement. The derivation of these inequalities assumes nearly nothing about the encountered correlation (except that the correlations concern 2-valued observables). It also clear in what way locality and causality (in the form of PI and OI) enter into the derivation. In chapter 5 we will see further reasons why these inequalities are more useful than Bell's original inequality.

As mentioned before the GHZ argument is tempting for its elegance, but on closer scrutiny it turns out that the assumptions necessary to obtain the result are quite strong, which makes the argument itself quite weak. The derivation uses a lot of properties of quantum mechanics. Specifically the link between eigenvalues of operators and the outcome of measurements and the assumption that these values satisfy the relations of the operators if these commute are in fact quite strong assumptions, which are themselves the subject of doubt. Therefore the GHZ argument, although elegant, adds very little to the discussion.

4 Bell's inequality and relativity

Can we use an entangled state to communicate? This is one of the main questions this chapter is about. In order to survey this subject, we first take a look at what quantum mechanics tells us, then we examine (superluminal) signals, ending with some notes on (superluminal) causation. For a more detailed account see Maudlin [29].

4.1 Signaling

The theory of relativity is often portrayed as one in which signals cannot be sent faster than the speed of light. But by what principle are we to define a signal? Energy, momentum and mass are notions that appear in our physical laws, but not signal. So before making such a strong statement about signals we need to know what a signal is.

We define a signal as follows: T(ransmitter) and R(eceiver) are two given systems, and we require that T has a controllable aspect that is correlated with an observable aspect of R. We are interested in a situation in which Bell's inequality is violated. In case of our example in paragraph 3.4.1 the controllable aspect would be the setting of the polarizer, for we cannot force the particle to give a specific answer. The observable aspect is the outcome of the measurement, since one cannot observe anything else about the state of the photons. So if you want to use this entangled system to send superluminal signals, the setting of the polarizer at one side must be correlated with the outcome of the measurement at the other side.

Let us have a look at our example of paragraph 3.4.1 once again. We want the behaviour of the photon at the receiver's end to depend on the polarizer setting of the other. We assume the photon at the transmitter's end reaches the polarizer 'first'. The chance it will pass is 50%. In case it passed, the other photon will pass with a chance of $\cos^2(\alpha)$. In case of absorption the other photon will pass with a chance of $1 - \cos^2(\alpha) = \sin^2(\alpha)$. So the the total chance that the photon at the receiving end will pass is always $50\% \cos^2(\alpha) + 50\% \sin^2(\alpha) = 50\%$. Therefore the receiver cannot extract any information out of his measurements.

One obvious flaw of the above argument is the notion of the 'first' photon. If the events in question are space-like separated (as they are), there always exist frames of reference in which the first event in our original frame is the second in the new frame. But one can say that in each frame of reference one event precedes the other and apply the argument to that frame to show that people living in that frame cannot use the entangled state to send signals. In the following discussion about compatibility of superluminal signals with relativity, we will even show that superluminal signals cannot exist at all. Before returning to our question about the compatibility of superluminal signaling with relativity, we first take a closer look at what quantum mechanics tells us about signaling.

To see what quantum mechanics says about signaling, we have to look at the objects that can be either controlled or observed. In quantum mechanics observables are self-adjoint operators on a Hilbert space. The controllable aspects of quantum mechanics are a bit harder to describe. We will focus on one aspect: the measurement. We arrive at the following question:

- Can a wave collapse in one wing produce an observable change in the distant wing?

The answer to this question is negative. We consider two spin-$\frac{1}{2}$ particles in the singlet state, which we denote by $\frac{1}{2}\sqrt{2}(\uparrow\downarrow - \downarrow\uparrow)$. When we measure the spin of the left particle, we will find spin up half of the time (and spin down the other half). In case of spin up the wave function will collapse to (↑↓). And since the chance that the outcome of the first measurement is spin up is precisely 50 percent, the wave collapses to (↑↓) precisely 50 percent of the time. So if you start with an ensemble of entangled spin-$\frac{1}{2}$ particles $\frac{1}{2}\sqrt{2}(\uparrow\downarrow - \downarrow\uparrow)$ and measure the spin of one particle, it will become an ensemble in which half of the systems are in the $(\uparrow\downarrow)$ state and the other half in $(\downarrow\uparrow)$ state. Therefore if you measure at the other side the chance of finding spin up is still 50 percent. This mixture is statistically equivalent to the original entangled states if you are only concerned about observables located at the wing, to which no measurement was performed.

In general, the state of a system of a number of(space-like) separated particles is an element of the tensor product of Hilbert spaces. A measurement of an observable of one particle (associated with one subspace) will lead to a mixture of (multi-particle) states that has exactly the same statistical implication for observables localized in other subspaces. Hence wave collapse cannot be used to send signals. The correlation only becomes clear if the measurement of two observables are brought together.

We return to our investigation of superluminal signals in relativity. A relativistic theory is in principle a theory that is Lorentz invariant. This implies that the speed of light is constant for every observer. A priori the speed of light is not a limit (for signals, energy transfer or anything else). Now let us take a look at what we can say about superluminal signals.

Suppose T sends a superluminal signal to R (in the rest frame of T). This signal will be superluminal for every observer (whose velocity is lower than c). If we assume that the signal leaves T before it is received by R in the rest frame of T, there will always be a frame in which the signal is received before it was sent. From a classical[10] point of view this is impossible, and in fact this statement is based on the following two principles: (1) The emission of a signal must come prior to its reception. (2) This equally holds for every observer, no matter what his speed. The rest of this paragraph is devoted to an examination of these two principles in combination with relativity theory.

As we have seen, if superluminal signals exist,then there are frames in which the signal is received before it was sent, so in that frame the effect precedes its cause. The question remains why this is objectionable.

[10]It is classical in the sense that the emission of the signal must come prior to its reception and it is relativistic because this property has to hold for every observer.

Maudlin [29] summarizes three types of superluminal signals. One in which the point of signal reception merely depends on the state of the transmitter, and two where the point of signal reception depends either deterministically or stochastically on the location of the transmitter only. He claims that these three types cover all possibilities. He concludes that none of these signals necessarily require a preferred Lorentz frame. So far there is no problem with relativity.

The reason superluminal signals will turn out to be impossible lies in the impossibility of signal loops. This is a construction in which a signal-chain can be received by the same system that emitted the first signal in the chain and the reception is earlier than the emission (on the clock of the emitter/receiver). As an example of an impossible signal loop, consider a superluminal telephone. In a certain frame of reference[11] Prime Minister Blair uses the phone to contact his colleague Bush, from whom he is space-like separated. Blair tells Bush one of his secret agents was assassinated in London half an hour ago. Blair spoke his words at 12:00 in the fixed reference frame and Bush receives those words at 11:00. Bush decides to call the secret agent (who was still alive at 11:00) and informs him about his upcoming assassination. Bush speaks his words at 11:10 and the secret agent receives them at 10:10. So the secret agent lives. Apart from him, Blair now of course decides not to send any signal. So Blair bases his decision to send a signal on a cause that was caused by his own signal. In terms of abstract signals: When superluminal signals are allowed, the decision to send a signal can depend on the reception of a signal caused by the first signal.

If we only allow superluminal signals to be sent from one point in space-time or from one worldline of a subluminal system, no signal loops can occur. Because the signal reception point does not lie in the past light cone of the transmitter and because only subluminal signals can be sent from the point of reception, the transmitter cannot be reached. The problems come when superluminal signals can be sent back and forth. As it turns out, all three types of signal imply signal loops are possible. We now have to choose between superluminal signals that allow loops and relativity. Because superluminal signals that do not allow loops cannot exist in relativity theory, for we would have to add extra structure to space-time, which will lead to an inconsistency with the principle of relativity.

We argued in this section that violations of Bell's inequality cannot be used to send signals. We have also seen in Chapter 3 that Bell's inequality cannot be violated by a theory in which all causes of some event lie in its past light cone. And now that we have found that superluminal signaling is impossible, we are left with the superluminal causation. We will explore this further in the next paragraph.

[11] All times given are with respect to this frame

4.2 Causation

The previous considerations showed that we cannot use a mechanism which violates Bell's inequality to send superluminal signals. In this section we will see that violation of Bell's inequality does lead to superluminal causation. The first part of this section deals with a search for a sufficient condition for superluminal causation. After that we will give an example that satisfies the condition and finally we examine whether these superluminal causal influences are compatible with relativity.

Some causal connections between two events distinguish themselves by the following property: "Had the first event been different, so would have the second." A causal connection, as opposed to the other possible explanations, supports this kind of counterfactuals. Not all causal connections satisfy this condition, as we demonstrate with the following example. Consider three events A, B and C. A causes C and B causes C. Even if A does not occur, C still can (due to B). So the above mentioned property is not true. But A definitely is a cause for C.

We now introduce the notion of two events being *causally implicated* with each other. Given two local events A and B, if the situation that A is different implies that B is different then A and B are causally implicated. 'Local' means that A and B can be considered as a point in space-time.

First, we make some remarks on the notion of causal implication. The statement, "A is causally implicated with B", does not imply that A caused B or vice versa. The possibility of a common cause is still left open. Another interesting feature of causal implication is that it can be symmetric. The case in which the cause is a necessary condition for the effect is an example of a symmetric causal implication. The effect occurs if and only if the cause does. Causation itself is, of course, not symmetric. Keeping this in mind, we would like to specify a sufficient condition for superluminal causation. Therefor we need to find a way to discount the counterfactual connection secured by common causes.

A sufficient condition for superluminal influences would be as follows:

(1) A and B are space-like separated local events.

(2) A would not have occurred if:

   a. B had not occurred,

   b. and everything in A's past lightcone had been unchanged.

This rules out the possibility of a common cause. If B does not occur, any common cause does not occur either, which is inconsistent with the unchanged past light cone of A. So common causes in A's past light cone are ruled out. To see that this is a sufficient condition for superluminal influences, note that in all cases of causal implication either A causes B, B causes A, or A and B have a common cause. In each case, superluminal influences play a role. In the first two cases this is obvious and in the third this is quite easily seen because the common cause cannot lie in the past light cone of A.

The next task is to find out whether violation of Bell's inequality implies that our condition is satisfied. Therefore, we need to appeal to our physical laws, because we want to know what would happen if something else did not happen. We cannot perform some kind of experiment to answer this question. But now we seem to be stuck, because we have not yet supposed that we know what the laws of nature are. We have only assumed that the laws predict the quantum correlations of photon pairs or a two particle spin-$\frac{1}{2}$ system in the singlet state. (Because we assume Bell's inequality to be violated.) So we consider two possibilities: (1) deterministic laws and (2) indeterministic laws. Although this paper is mostly about quantum mechanics, which is an indeterministic theory, we will consider both cases, because with very little extra effort the result improves remarkably if we do so.

In a deterministic theory the laws fix a unique state for every system at any moment in time, given certain initial conditions. One of the main differences with quantum mechanics is measurement. This evolves without a stochastic element. In this deterministic case the result of our observation (for example of our spin measurement) is determined by the laws of our theory together with some physical facts. Some of these physical facts must be at space-like separation, because if they were not Bell's inequality would hold (and we assumed it did not). So in some cases the result on one side depends on the distant setting. In terms of counterfactuals this means that in some cases the result of the measurement would have been different if the distant setting had been different. And the distant setting does not require any change in the past light cone of the local measurement. So according to our sufficient condition this is an example of superluminal causation.

Our next step is the indeterministic case. This means that at least one of the two measurements involves stochastic elements. Unlike the deterministic case, we only consider the situation in which the polarizers are aligned (or the spin is measured in the same direction). Since the laws imply perfect correlation, they also support the counterfactual claim that had one photon not passed, neither had the other. We only have to rule out the common cause option. This is quite simple. Because the process on one of the sides is stochastic, the outcome could have been different without any change in its past light cone. So a common cause explanation is impossible. Consequently the two sides are causally implicated with each other.

In fact, this is not the whole story, as there are some objections (e.g. the difference between 'causally implicated' and 'causation') to call the quantum connection causal, but these can all be dismissed (see Maudlin [29]). With that in mind, we conclude that violation of Bell's inequality implies superluminal causation. Our last question of this paragraph is to examine whether superluminal causation is compatible with relativity.

In a non-relativistic world the photon experiment is simple. The outcome of the first measurement is the cause of the outcome of the second.[12] In relativity theory this reasoning goes wrong, because the events are space-like separated and an absolute notion of time order does not exist. There are two possible ways of responding to this situation. One is just to refuse to designate one event as a cause. Just say that the events are simply causally connected, and that is it. The other is to say the cause depends on the frame of reference. This can be done, because within one frame of reference it is easy to say which event was earlier.

The objection against superluminal signaling was the impossibility of signal loops. Why can't they occur with causal connections? If we try to repeat the argument of signal loops for causal loops, we get (1) superluminal causation implies backwards causation in some frames of reference, (2) this backward causation leads to systems with unacceptable outcomes, (a plane could decide to take off if and only if it has not touched down). (3) Since this cannot be true one of our assumptions is wrong and in fact superluminal causation is the main suspect.

This argument fails at every step. Of course we only need to show one failure for our purposes. We choose the first. Superluminal causation does not imply that any identifiable cause precedes an identifiable effect, as we have seen. This argument shows that no causal loops exist.

Finally, we need to show that superluminal causal connections do not pick out a preferred Lorentz frame, in order to assure compatibility between the connection and the principle of relativity. The discussion of the previous paragraph about compatibility of superluminal signals can be repeated for superluminal causation. But in this case we do not have to choose between relativity and superluminal causation that allows causal loops, because those loops are impossible. So superluminal causation does not give rise to inconsistency with the theory of relativity.

4.3 Conclusion

Summarizing the results of the two paragraphs above:

- Relativity allows no superluminal signals.
- Violation of Bell's inequality implies superluminal causation.

[12] In principle simultaneous measurements are possible which would lead to problems, but the chance of the two measurements being exactly simultaneous is zero.

5 Experimental tests of Bell's inequality

In the following pages we will be discussing experimental tests of Bell's inequality. Or, more precisely, experimental tests of a slightly modified version of Bell's inequality, known as the Clauser-Horne-Shimony-Holt inequality, which is more suitable to test experimentally. A detailed review of various relevant experiments will be presented; but not before we have done some preliminary work. We will discuss locality in some more depth and see that it is one of the important things one has to be aware of while experimenting.

5.1 On the locality condition

Our starting point is the Clauser-Horne inequalities as we derived them in section 3.6. The crucial assumptions we made to derive them were outcome independence (OI) and parameter independence (PI). The former expressed that the outcome of a measurement in one system does not influence the outcome of a measurement in the other system. The latter expressed that the outcome of a measurement in one system does not depend on the choice of measurement in the other.

If in any experiment we find a violation of the Clauser-Horne inequalities we know that one of these assumptions is false. The challenge when doing an experiment to test these inequalities is to create a setup wherein influence of one measurement of an other through classical means (i.e. by signaling of some sort) is impossible. For the OI condition this is satisfied when both measurements are carried out simultaneously, which poses little difficulty. More troublesome is the PI condition. To exclude the possibility that PI violation is caused by signaling, the choice of one measurement should take place at a moment in space-time with space-like separation to the other measurement and vice versa. It turns out that in practical situations this is very difficult.

The earliest experiments, which featured a static choice of measurements that could be changed between runs, did not satisfy this condition. Although it seems unlikely that the static choice of measurement could influence the outcome of the measurements at some other distant location (our physical intuition tells us that this should not be the case and therefore that PI is a reasonable assumption), there is no physical argument that forbids such an influence. This therefore was a major point of critique for the earlier static experiments.

Even later experiments that feature a setup capable of changing the choice of measurement for each measurement do not fully satisfy the PI requirements, because at best the choice of measurement has been pseudo random allowing for the slim possibility that some knowledge of the choice of measurement reaches the other measurement. This therefore continues to be a point of critique for the experiments.

5.2 From Bell's theorem to a realistic experiment

Although Bell's inequality (3.22) provides the possibility of doing a decisive experimental test of the entire family of local hidden variable theories, it is not in a form that can be tested directly in a realizable experiment.

$$|q(\mathbf{a},\mathbf{b}) - q(\mathbf{a},\mathbf{c})| \leq 1 + q(\mathbf{b},\mathbf{c}). \tag{3.22}$$

Experiments testing Bell's inequality usually are of the following form: there is a source emitting pairs of photons[13] in a correlated state. The photons move in opposite directions towards two linear polarizers, which can have different orientations. If the photons pass the polarizers they are detected by photon detectors.

One problem that occurs when such a setup is used to test Bell's inequality is caused by the fact that the photon detectors have a limited efficiency. The setup cannot distinguish whether a photon was absorbed by the polarizer or that it was simply missed. It is therefore impossible to directly determine the correlation functions featured in Bell's inequality.

The Clauser-Horne (CH) inequalities already form an enormous improvement since they hold for any correlated events, for which the correlation can be explained by a local hidden variable theory. Even though the detection is not 100 percent efficient the probabilities of detecting a photon for each of the detectors are still correlated. So the CH inequalities do still apply.

Though the CH inequalities are much more suitable for experimental test than Bell's inequality, there still are a number of practical problems, which need resolving.

(a) Clauser and Horne showed that the efficiencies that could be obtained at the time were too low to be able to show violations of their inequalities by quantum mechanics. [11]

(b) The CH inequalities feature both single event and joint probabilities. The experimental setups usually only measure double detections. Even if one of the polarizers is removed the single event probability cannot be effectively measured because the limited efficiency of both detectors still plays a role.

To resolve (a) Clauser and Horne made an additional assumption that leads to a stronger inequality:

> ... for every emission of a photon in a state λ, the probability of a count with a polarizer in place is less than or equal to the probability with the polarizer removed. (*no enhancement hypothesis*)

With this assumption both problems can be resolved. If we introduce the notation $p_\lambda(\infty_a)$ for the probability of a count with the left detector (L or A side) when a photon is emitted in

[13] Photons are used because this makes it easier to satisfy the locality conditions mentioned above.

(hidden) state λ and the polarizer is removed, the CH assumption implies that

$$\begin{aligned} p_\lambda(a), p_\lambda(a') &\leq p_\lambda(\infty_a), \\ p_\lambda(b), p_\lambda(b') &\leq p_\lambda(\infty_b), \end{aligned} \tag{5.1}$$

taking K as defined in equation (3.38) and

$$\begin{aligned} \alpha &= p_\lambda(a'), & \beta &= p_\lambda(b), \\ \alpha' &= p_\lambda(a), & \beta' &= p_\lambda(b'), \\ A &= p_\lambda(\infty_a), & B &= p_\lambda(\infty_b), \end{aligned}$$

one obtains analogously to the derivation of the original CH inequalities

$$0 \leq p(a' \& \infty_b) + p(\infty_a \& b) + p(a \& b') - p(a' \& b) - p(a \& b) - p(a' \& b') \leq p(\infty_a \& \infty_b). \tag{5.2}$$

Rewriting this one finds the strong version of the Clauser and Horne inequalities

$$\boxed{-1 \leq S \leq 0,} \tag{5.3}$$

where

$$S = \frac{p(a \,\&\, b) - p(a \,\&\, b') + p(a' \,\&\, b) + p(a' \,\&\, b') - p(a' \,\&\, \infty_b) - p(\infty_a \,\&\, b)}{p(\infty_a \,\&\, \infty_b)}. \tag{5.4}$$

5.3 Experiments: an overview

As the reader will know by now, with Bell's theorem, the debate on the possibility (or necessity) of completing quantum mechanics changed from one of personal philosophical taste, to one in which experiments can settle the question. Choosing a situation where quantum mechanics predicts that Bell's inequality is violated, we have a test that allows us to discriminate between quantum mechanics and any local hidden variable theory.

As we have seen Bell's inequality (or more specifically the strong version of the Clauser-Horne inequalities) provides us with an experimental test to settle the EPR debate. To prove whether Nature violates the causality and locality conditions expressed by parameter and outcome independence we need to find a situation, in which quantum mechanics predicts a violation of the CH inequalities. However, such situations in which a conflict arises are rare. In fact, Bell's inequality is compatible with classical physics. Relativistic mechanics and electrodynamics neatly satisfy Einstein's causality. So for violations we need to look in the quantum domain. Moreover, in the quantum mechanical framework we can point out two necessary conditions to have a conflict with Bell's inequalities: 1. The two separated (sub)systems have to be in an entangled (non-factorisable) state; 2. For each (sub)system, it must be possible to choose the measured quantity among at least two non-commuting observables (such as polarizer measurements along directions a and a'). And even in such cases the conflict only exists for well chosen sets of orientation.

Around 1965 it was realized that there was no experimental evidence of a violation of Bell's inequality available. It was desirable to design an experiment where the predictions made by quantum mechanics violate Bell's inequality. Pairs of photons emitted in suitable atomic radioactive cascades are good candidates for such a sensitive test.

5.3.1 First generation experiments ('seeding work')

Already in 1969, Clauser, Horne, Shimony, and Holt [12] had shown that it was possible to do a 'sensitive' experiment with correlated photons produced in certain atomic cascades. Two groups started an experiment (one in Berkeley and one in Harvard). Their results were conflicting. Some years later, a third experiment was carried out in Houston, Texas. But it was not very convincing. (The reader who is interested in the original papers can find a list of references in [1].)

All three used an experimental scheme, that was different (simplified) from the ideal one; it involved one-channel polarizers. A one-channel polarizer transmits light polarized parallel to a (or b), but blocks the orthogonal. One can thus only detect one of the results. In order to recover the missing data, extra runs are performed with one or both polarizers removed. One gets equation (5.3). For a suitable choice of angles this generalized Bell inequality is violated and therefore it is possible to make a sensitive test with one-channel polarizers also. As we stressed during the derivation of it, a supplementary assumption is required to get (5.3). This additional assumption is not unreasonable. The presence of a polarizer should not increase the chance of detecting a photon. But, still, it is an addition assumptional which can turn out to be false.

In the Berkeley experiment calcium atoms were excited. The signal was weak. It took more than 200 hours of measurement for a significant result. The results were in agreement with quantum mechanics and violated Bell's inequality (5.3) by 5 standard derivations.

At the same time in Harvard a result in disagreement with quantum mechanics was found, and in agreement with Bell's inequality. Their source was based on a cascade of Mercury 200, excited by electron bombardment. Data accumulation time: 150 hours. The same experiment, subsequently repeated, but with the 202 isotope of Mercury, gave an agreement with quantum mechanics, and a significant violation of Bell's inequality.

In Houston, in 1976, a much improved source of correlated photons was built (using another cascade in Mercury 200). They used a *C.W.* single line tunable laser (*C.W.* stands for continuous wave) to selectively excite the upper level of the cascade. Such lasers are nowadays not so special, but at that time they were quite rare. The signal was several orders of magnitude larger than in previous experiments. The result was in excellent agreement with quantum mechanics, and violated Bell's (adjusted) inequality by 4 standard deviations.

5.3.2 Orsay experiments (1980-1982)

All experiments done before the 'Orsay experiments' performed by Alain Aspect *et al.*, [2, 3] made use of static setups: setups in which polarizers were held fixed during the whole run. Although it is improbable that 'something strange' that could cause the correlations would happen (as a result of the detectors and the photons in question not having a completely separate history) there is no law in physics that forbids it. As Bell put it in [4]:

> . . . the settings of the instruments are made sufficiently in advance to allow them to reach some mutual rapport by exchange of signals with velocity less than or equal to that of light. In that connection, experiments of the type proposed by Bohm and Aharonov [8], in which the settings are changed during the flight of the particles, are crucial.

This nowadays famous Aspect-Dalibard-Roger (ADR) experiment was the first direct experimental test of Bell's inequality; the first in the sense that for the first time the settings were changed during the flight of the particles. In this experiment, a high-efficient stable calcium source emits a pair of (entangled) photons.

The polarizers that where used involved a switching device followed by two polarizers in two different orientations, a and a' on one side and b and b' on the other.

The optical switching was achieved by a Bragg reflection from an ultrasonic standing wave in water. The incident angle was equal to the Bragg angle, $\theta_B = 5 \times 10^{-3}\ \mathrm{rad}$. The light is either transmitted straight ahead or deflected at an angle $2\theta_B$. For a zero amplitude of the ultrasonic standing wave (which occurs twice during an acoustical period) the light is transmitted straight ahead. A quarter of a period later the amplitude of the standing wave is maximum; light is then (almost) fully reflected.

From the following experimental numbers it is clear that detection events are separated by a space-like interval. Switching between the two channels occurs each 10 ns. This delay is smaller than the time it takes the photons to reach the detectors $(12\ \mathrm{meter}/c \simeq 40\ \mathrm{ns})$. Also the lifetime (5 ns) of the intermediate level of the cascade used to generate the correlated photon pairs, is small compared to 40 ns. Therefore the detection event on one side and the corresponding change of orientation on the other side are separated by a space-like interval. Assuming EPR's locality principle the measurement on the distant polarizer cannot be influenced by the setting of the first polarizer. This is a crucial condition to show that Nature violates Bell's inequality.

The divergence of the beams had to be reduced in order to get good switching, which resulted in lower efficiencies, than in previous experiments. Finally Aspect, Dalibard, and Roger conclude that the random delayed choice scheme was not truly achieved because the changes were not random but quasi-periodic. However it was assumed that they functioned in an uncorrelated way, since the switches on the two sides were driven by different generators at different frequencies.

The most important result of the ADR experiment is an experimental value for S. The following experimental value was found for an angle of $\pi/8$:

$$S_{exp} = 0.101 \pm 0.020$$

,

violating the upper limit of Bell's inequality (5.3) by 5 standard deviations. It is in good agreement with the quantum mechanical prediction

$$S_{qm} = 0.113 \pm 0.005.$$

For other angles the results were also in good agreements with quantum mechanics. However, it should be noted that the experiment was not ideal. An important point is that single polarizers were used. As a consequence there was no way to know whether a photon missed the detector or whether it was blocked by the polarizer. To come any further assumptions have to be made. One usually assumes that the ensemble of detected pairs is a fair sample of the ensemble of all emitted pairs ("fair sampling assumption"). From this fair sampling assumption it follows that the ensemble of actually detected pairs is independent of the orientations of the polarizers. Besides this we also have the important imperfection of the 'random' switching, which in fact is 'quasi'-periodic.

Franson [17] points out that the ADR experiment does not rule out a class of theories in which the outcome of an event is undetermined until some time after its occurrence. This class of theories includes not only quantum mechanics, but also various local, realistic theories as well. From a classical point of view, the optical switches and measuring devices must determine whether or not a photon has been absorbed in a given detector at the same instant that such an event would have occurred. But if the outcomes of such events are not actually determined until some later time, then information regarding the orientations of the polarizers could conceivably be exchanged at velocities less than that of light and used during the subsequent determination process. Although such theories may seem counterintuitive, this should not preclude their consideration. To conclude Franson's point is that it is not evident that the ADR experiment rules out all local theories in which the outcome of an event is not determined until after it's apparent time of occurrence.

Until 1998 the ADR experiment was the only experiment involving fast changes of the settings of the analyzers. Imperfections still left open the possibility of ad hoc supplementary parameter models fulfilling Einstein causality.

5.4 Towards an ideal experiment

The reason that not much happened during such a long period was probably the fact that the cascade used by Aspect et al. was such that there was not much room left for improvements with sources based on atomic radiative cascades. In an ideal experiment the loophole related to low detector inefficiency should be closed. Also, ideally, one needs polarizers that can be independently reoriented at random times in short time (shorter than L/c). An experiment of this type ('ideal timing experiment') has been completed in 1998 by the group of Anton Zeilinger [40]: "Violation of Bell's inequality under strict Einstein locality conditions". However the detector inefficiency loophole remained, but it is believed that this 'limited efficiency of detectors loophole' can be closed in the foreseeable future by a technological advance. It does not correspond to a radical change in the experimental scheme. Nonetheless, such an experiment is highly desirable.

Maybe less important, but interesting to mention, is that very recently (June 2004) an experimental violation of Bell's inequality was reported for a single atom and a single photon [34], being the first demonstration of the violation with particles of different species.

6 C*-algebras

In this chapter we will approach the subject of hidden variables from a much more general perspective. In the previous chapters we limited ourselves to the very specific situation of systems of two or more spin-$\frac{1}{2}$ particles or similar systems. We will now examine if similar results may be obtained for more general systems.

We will do so by exploiting the operator-algebraic framework of quantum mechanics. In this formalism the starting point is a C*-algebra of operators. The states can then be derived from this algebra. In fact, all properties of a theory are supposed to be contained in the algebraic structure. Specifically, it turns out that the existence of hidden variable theories and the violation of Bell's inequality is related to the possible commutativity of the algebra. Obviously, this path requires some knowledge of C*-algebras, a subject that many physicists know very little of. To maintain accessibility for a broader group of people the first section will be dedicated to the basic theory of C*-algebras.

6.1 C*-algebra basics

A complete treatment of the theory of C*-algebras is far beyond the scope of this document. Even a rudimentary introduction would be a project on its own. We will therefore just give the definitions and fundamental results of the theory without any proof. For a more detailed treatment of the subject we refer to the lecture notes of our supervisor Klaas Landsman on the subject. They have been of great help to us in understanding the subject, and are available as math-ph/9807030 from the arXiv[28].

A C*-algebra is basically a generalization of the algebra of bounded linear operators on a Hilbert space. It is a normed vector space over $\mathbb{C}$ that has a product and an operation called involution, denoted by *. The former behaves as the composition of operators, the latter is the generalization of taking the adjoint of an operator.

We now give a more exact definition. We will start by defining associative algebras.

Definition 6.1. *An **associative algebra** is a vector space $\mathfrak{A}$ with an additional operation $\cdot : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$, called the product that satisfies the following conditions:*

A1: $A(BC) = (AB)C = ABC \quad \forall A, B, C \in \mathfrak{A}$,

A2: $(\lambda A + \mu B)C = \lambda AC + \mu BC$ *and* $C(\lambda A + \mu B) = \lambda CA + \mu CB$
$\forall A, B, C \in \mathfrak{A}$ *and* $\forall \lambda, \mu \in \mathbb{C}$.

If we add an involution we get a *-algebra.

Definition 6.2. *A ***-algebra** is an associative algebra $\mathfrak{A}$ with an additional operation $* : \mathfrak{A} \longrightarrow \mathfrak{A}$, called involution that satisfies the following conditions:*

I1: $(A^*)^* = A \quad \forall A \in \mathfrak{A}$,

I2: $(AB)^* = B^* A^* \quad \forall A, B \in \mathfrak{A}$,

I3: $(\lambda A)^* = \overline{\lambda} A^* \quad \forall A \in \mathfrak{A}; \forall \lambda \in \mathbb{C}$.

We can now give the definition of a C*-algebra.

Definition 6.3. *A* ***C*-algebra*** *is a *-algebra $\mathfrak{A}$ on which a norm $\|\cdot\| : \mathfrak{A} \longrightarrow \mathbb{R}$ is defined that satisfies the standard conditions of a norm:*

N1: $\|A\| \geq 0 \quad \forall A \in \mathfrak{A}$,

N2: $\|A\| = 0 \Leftrightarrow A = 0$,

N3: $\|\lambda A\| = |\lambda| \, \|A\| \quad \forall A \in \mathfrak{A}; \forall \lambda \in \mathbb{C}$,

N4: $\|A + B\| \leq \|A\| + \|B\| \quad \forall A, B \in \mathfrak{A}$.

And $\mathfrak{A}$ satisfies the following conditions:

1. *$\mathfrak{A}$ is complete in the norm $\|\cdot\|$. (If for a sequence $\{A_n\}$ one has $\|A_n - A_m\| \longrightarrow 0$ when n and m go to infinity, then $\{A_n\}$ has a limit in $\mathfrak{A}$.)*
2. $\|AB\| \leq \|A\| \|B\| \quad \forall A, B \in \mathfrak{A}$
3. $\|A^* A\| = \|A\|^2 \quad \forall A \in \mathfrak{A}$

A subset $\mathfrak{B}$ of a C-algebra $\mathfrak{A}$ that itself is a C*-algebra with the operations of $\mathfrak{A}$ is called a* ***C*-subalgebra****.*

Besides the standard axioms presented above, a C*-algebra may also have the following properties:

Definition 6.4. *A C*-algebra $\mathfrak{A}$ is called* ***commutative****, if for all $A, B \in \mathfrak{A}$ it satisfies:*

$$AB = BA.$$

Definition 6.5. *A C*-algebra $\mathfrak{A}$ is called* ***unital*** *if there exists an $\mathbb{I} \in \mathfrak{A}$ satisfying:*

$$A\mathbb{I} = A = \mathbb{I}A \quad \forall A \in \mathfrak{A}.$$

In what follows for reasons of simplicity it is assumed that all C*-algebras have a unit, unless explicitly stated otherwise. Most results may be generalized to non-unital C*-algebras by embedding into a C*-algebra with unit. Such an embedding always exists, and is unique.

6.1.1 Self-adjoint and positive elements

The concept of self-adjointness of bounded linear operators has a natural generalization for C^*-algebras. Recalling that the involution operator was defined to behave like taking the adjoint of an operator, the following definition will seem very natural.

Definition 6.6. *An element A of $\mathfrak{A}$ is called **self-adjoint** if $A = A^*$. The collection of self-adjoint elements in $\mathfrak{A}$ is denoted by:*

$$\mathfrak{A}_{\mathbb{R}} := \{A \in \mathfrak{A} | A = A^*\}.$$

If for an $A \in \mathfrak{A}$ there is a $B \in \mathfrak{A}$ such that $AB = \mathbb{I} = BA$, then A has an inverse in $\mathfrak{A}$. This concept may be used to generalize the notion of the spectrum of an operator.

Definition 6.7. *The **spectrum** $\sigma(A)$ of $A \in \mathfrak{A}$ is the set*

$$\sigma(A) := \{z \in \mathbb{C} | A - z\mathbb{I} \text{ has no inverse}\}.$$

As with linear operators the spectrum of a self-adjoint element contains only real numbers. One can also generalize the notion of positivity of operators.

Definition 6.8. *An element A of $\mathfrak{A}$ is called **positive** if it is self-adjoint and its spectrum is positive, that is:*

$$\sigma(A) \subset \{r \in \mathbb{R} | r \geq 0\}.$$

The set $\mathfrak{A}^+$ of all positive elements of $\mathfrak{A}$ may be characterized as follows:

Theorem 6.9.

$$\begin{aligned} \mathfrak{A}^+ &= \{A^2 | A \in \mathfrak{A}_{\mathbb{R}}\} \\ &= \{B^* B | B \in \mathfrak{A}\}. \end{aligned}$$

The following two lemmas describe properties of positive elements that will prove useful in proving certain results.

Lemma 6.10. *If A and B commute, then A is positive and B is positive implies AB is positive.*

Lemma 6.11. *Any self-adjoint $A \in \mathfrak{A}$ can be written as $A = A_+ - A_-$, with A_+ and A_- positive, $A_+ A_- = 0$, and $\|A_\pm\| \leq \|A\|$.*

A map that conserves the property of positivity is called positive:

Definition 6.12. *A linear map $L : \mathfrak{A} \longrightarrow \mathfrak{B}$ between two C^*-algebras is called **positive** when it maps $\mathfrak{A}^+$ to $\mathfrak{B}^+$.*

6.1.2 States

The concept of a state plays an important role in quantum mechanics. It is defined in terms of the operators only and therefore it may be generalized to C*-algebras, in whose theory it plays a central role.

Definition 6.13. *A* ***state*** *on a C*-algebra is a linear map $\phi : \mathfrak{A} \longrightarrow \mathbb{C}$ that satisfies:*

$$\phi(A) \geq 0 \quad \forall A \in \mathfrak{A}^+,$$

and

$$\phi(\mathbb{I}) = 1$$

. The state space is denoted $\mathcal{S}(\mathfrak{A})$.

Some properties of states, which we will need for some of the proofs in this chapter, are given by the following lemma:

Lemma 6.14. *Let ϕ be a state on a C*-algebra $\mathfrak{A}$. Then*

1. $|\phi(A^*B)|^2 \leq \phi(A^*A)\phi(B^*B) \quad \forall A, B \in \mathfrak{A}$.
2. $\phi(A^*) = \overline{\phi(A)} \quad \forall A \in \mathfrak{A}$.
3. *If $A = A^*$, then there is a state ϕ such that $|\phi(A)| = \|A\|$.*

An important property of the state space $\mathcal{S}(\mathfrak{A})$ is the following:

Theorem 6.15. *The state space $\mathcal{S}(\mathfrak{A})$ is a* ***convex set****. Thus if ϕ and ψ are states, then $\lambda\phi + (1-\lambda)\psi$ with $\lambda \in [0,1]$ is also a state.*

The pure states may be defined as the extreme points of the state space.

Definition 6.16. *A* ***pure state*** *is a state $\phi \in \mathcal{S}(\mathfrak{A})$ for which:*

$$\phi = \lambda\phi_1 + (1-\lambda)\phi_2,\ \lambda \in (0,1) \Rightarrow \phi = \phi_1 = \phi_2.$$

Definition 6.17. *The* ***dispersion*** *of a state $\phi \in \mathcal{S}(\mathfrak{A})$ on an element $A \in \mathfrak{A}$ is defined as:*

$$\sigma_\phi(A) \equiv \phi(A^2) - \phi(A)^2.$$

One of the most prominent properties of quantum mechanics is the Heisenberg uncertainty principle. Using the generalized concepts presented above, the principle also applies to general C*-algebras.

Theorem 6.18. *For any state $\phi \in \mathcal{S}(\mathfrak{A})$ and any two elements $A, B \in \mathfrak{A}_\mathbb{R}$ the following inequality holds:*

$$\sigma_\phi(A)\sigma_\phi(B) \geq \frac{1}{4}\left|\phi([A,B])\right|^2,$$

where $[A, B]$ is the commutator $AB - BA$.

6.1.3 Commutative C^*-algebras

The space $C(X)$ of continuous functions on a compact Hausdorff space X forms a C^*-algebra in a natural way. The space is complete in the supremum norm:

$$\|f\|_\infty := \sup_{x\in X} |f(x)| \, .$$

The addition, product and scalar product operations are defined in the usual way. Taking the complex conjugate of a function defines an involution. One may easily check that all axioms of definition 6.3 hold. In fact, the C^*-algebra is also commutative and has a unit.

The following theorem states, that the converse is also true; all commutative unital C^*-algebras may be identified with some space of continuous functions.

Theorem 6.19. *Let $\mathfrak{A}$ be a commutative C^*-algebra with unit. Then there is a compact Hausdorff space X such that $\mathfrak{A}$ is isometrically isomorphic to $C(X)$. This space is unique up to homeomorphism.*

The conditions may be stretched a little so that the statement becomes true for locally compact Hausdorff spaces (like $\mathbb{R}^3$). In that case we must consider a slightly smaller class of functions namely the class of continuous functions that *vanish at 'infinity'*. This means that for any function f there are compact subsets of X outside of which f becomes arbitrarily small. This space (denoted by $C_0(X)$) again is a commutative C^*-algebra, but this time without a unit. Theorem 6.19 becomes:

Theorem 6.20. *Let $\mathfrak{A}$ be a commutative C^*-algebra. Then there is a locally compact Hausdorff space X such that $\mathfrak{A}$ is isometrically isomorphic to $C_0(X)$. This space is unique up to homeomorphism.*

The following theorem gives a characterization of the states on a commutative C^*-algebra.

Theorem 6.21. *The state space of the commutative C^*-algebra $\mathfrak{A} = C(X)$ consists of all probability measures on X. The pure states are Dirac measures, and can therefore be identified with the points of X.*

The above characterization of commutative C^*-algebras gives us the following sufficient condition for the positivity of an element.

Theorem 6.22. *Let $\mathfrak{A}$ be a commutative C^*-algebra. For all $A \in \mathfrak{A}$ the following equivalency holds.*

$$A \text{ is positive} \Leftrightarrow \phi(A) \geq 0 \quad \forall \phi \in \mathcal{S}(\mathfrak{A}).$$

6.1.4 Representations

Definition 6.23. *A **representation** of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$ is a complex linear map $\pi : \mathfrak{A} \longrightarrow \mathfrak{B}(\mathcal{H})$ satisfying*

$$\begin{aligned} \pi(AB) &= \pi(A)\pi(B), \\ \pi(A^*) &= \pi(A)^* \end{aligned}$$

for all $A, B \in \mathfrak{A}$

Since the definition of a C*-algebra was formulated to reflect some of the properties of the algebra of bounded operators on a Hilbert space $\mathfrak{B}(\mathcal{H})$, it is no surprise, that such representations are possible for some C*-algebras. (The identity map on $\mathfrak{B}(\mathcal{H})$ is of course a very trivial example of a representation.) What may come as a surprise, though, is that any C*-algebra may be represented in this way.

Theorem 6.24. ***(Gel'fand-Neumark)*** *A C*-algebra is isomorphic to a *-subalgebra of $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

6.1.5 Von Neumann algebras

In the light of the Gel'fand-Neumark theorem, any C*-algebra may be viewed as a *-subalgebra of $\mathfrak{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. The completeness of the C*-algebra in fact also tells us that the *-subalgebra is closed in the standard norm-topology. For quantum mechanics we sometimes need a little a more than this, and wish to consider C*-subalgebras of $\mathfrak{B}(\mathcal{H})$, which are closed in a weaker topology.

Definition 6.25. *Let $\mathcal{H}$ a Hilbert space.*

- *The **norm-topology** on $\mathfrak{B}(\mathcal{H})$ is the topology defined by the operator norm*

$$\|A\| := \inf\{C \in \mathbb{R} | \forall \psi \in \mathcal{H} : \|A\psi\| \leq C\|\psi\|\}.$$

- *The **strong (operator) topology** on $\mathfrak{B}(\mathcal{H})$ is defined by the property that $A_n \longrightarrow A$ if and only if $\|(A_n - A)\psi\| \longrightarrow 0$ for all $\psi \in \mathcal{H}$.*

- *The **weak (operator) topology** on $\mathfrak{B}(\mathcal{H})$ is defined by the property that $A_n \longrightarrow A$ if and only if $|\langle\psi, (A_n - A)\psi\rangle| \longrightarrow 0$ for all $\psi \in \mathcal{H}$.*

The norm-topology is stronger (less coarse) than the strong operator topology, which in turn is stronger (as the name implies) than the weak operator topology. A fundamental result shows that the statements that a unital *-subalgebra of $\mathfrak{B}(\mathcal{H})$ is closed in the strong and that it is closed in the weak operator topology, are equivalent. Such algebras are also called **von Neumann algebras**.

6.2 Classical and quantum mechanics

In classical mechanics the (pure) states of a system are represented by points in some phase space X. Arbitrary (statistical) states are represented by probability measures on X. The observables are real valued functions on X. The pure states of the system are dispersion free.

With the additional assumptions that X is locally compact and Hausdorff and that the observables are continuous functions that go to zero at infinity, the space of observables becomes a commutative C*-algebra. Note that the extra conditions made here are required to satisfy the axioms of a C*-algebra and do not affect the commutativity.

The converse is not necessarily true. A commutative C*-algebra in general cannot be represented by a phase space theory. This would require extra structural properties, which are not guaranteed by the structure of a commutative C*-algebra.

We may have concluded from section 6.1.3 that any theory represented by a noncommutative C*-algebra is non-classical (i.e., it may not be expressed as a phase space theory). The Heisenberg uncertainty relation implies that all these theories share the property, that they have no dispersion-free states. It is quite clear that quantum mechanics fits in this category of theories.

This very simple characterization of classical and non-classical theories by an algebraic property of C*-algebras raises some interesting possibilities. Namely, the question whether some of the difficulties concerning quantum mechanics can be solved through classical means, may be investigated by looking at the algebraic properties of the C*-algebras involved. This will be done in the following sections.

6.3 Hidden theories in C*-algebras

Arguably, the most common response when one first encounters the statistical nature of the quantum mechanical state is the idea that this statistical nature is caused by our ignorance. Consciously or subconsciously, this idea is fuelled by our experiences with classical physics, specifically with statistical mechanics. In statistical mechanics we too have states that have a statistical nature. These states represent an ensemble of states in which a system might be, one of which is the true state of the system. The statistical nature of these states is caused by the fact that we do not know which state is the true state of the system. One is tempted to think that the same might be true for the states in quantum mechanics, that these states too might represent ensembles of states with no statistical nature, but as yet unknown to us. This is the basic thought behind the idea of hidden variables.

The matter of the existence of hidden variables in quantum mechanics has been debated on and off for over the last century. Over the years the concept of hidden (variable) theory has broadened a little. We no longer require the states of a hidden theory to be non-statistical. We only require them less statistically uncertain. This, of course, is a very vague statement and in any systematic treatment of the problem we will have to specify what we mean by this.

In this section we exploit the C*-algebraic framework of quantum mechanics to investigate the problem of the existence of hidden variable theories in quantum mechanics. To do so we will need an exact definition of a hidden variable theory in the language of C*-algebras. Redei gives the following definition in [38].

Definition 6.26. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two unital C*-algebras (representing physical theories). $\mathfrak{B}$ is said to be a* ***hidden theory*** *of $\mathfrak{A}$ with respect to some map $L : \mathfrak{B} \longrightarrow \mathfrak{A}$ if the following conditions hold.*

1. *L is a linear, positive, unit preserving map.*
2. *For all $\phi \in \mathcal{S}(\mathfrak{A})$ there is a Borel measure μ on $\mathcal{S}(\mathfrak{B})$ such that*
   (a) $\phi(B) = \int_{\mathcal{S}(\mathfrak{B})} \omega(B) d\mu(\omega) \quad \forall B \in \mathfrak{B}$.
   (b) *For every $B \in \mathfrak{B}$ for which LB is self-adjoint and $\sigma_\phi(LB) > 0$ we have*
   $$\sigma_\phi(B) > \sigma_\omega(B) \quad \forall \omega \in \mathrm{supp}(\mu);$$
   (c) *For every $B \in \mathfrak{B}$ for which LB is self-adjoint and $\sigma_\phi(LB) = 0$ we have*
   $$\sigma_\phi(B) = \sigma_\omega(B) = 0 \quad \forall \omega \in \mathrm{supp}(\mu).$$

We have seen in section 6.1.3 that the states of a commutative C*-algebra are probability measures and that the pure states are Dirac measures. Since any probability measure may be decomposed into Dirac measures, any state may be decomposed into dispersion-free states. We therefore have that any commutative C*-algebra is a dispersion-free hidden theory of itself.

It can be proven that the converse statement is also true (see [38] Proposition 9.4). This strengthens our assertion that commutative C*-algebras can be identified with classical theories.

For certain types of the map L, negative results for the existence of dispersive hidden theories may be obtained (see [38] section 9.2.2). We here look at the specific case where L is a conditional expectation.

Definition 6.27. *Let $\mathfrak{B}$ be a C*-algebra and $\mathfrak{A} \subset \mathfrak{B}$ a C*-subalgebra containing the unit of $\mathfrak{B}$. A positive linear map $L : \mathfrak{B} \longrightarrow \mathfrak{A}$ is called a* ***conditional expectation*** *if*

$$L(A_1 B A_2) = A_1 L(B) A_2 \quad \forall A_1, A_2 \in \mathfrak{A}; B \in \mathfrak{B}.$$

Suppose Q is a projection on some Hilbert space $\mathcal{H}$, then $L(A) = QAQ$ defines a positive linear map $L : \mathfrak{B}(\mathcal{B}) \longrightarrow \mathfrak{B}(\mathcal{B})$. One may easily check that L satisfies the conditions of a conditional expectation. In fact, the conditions for a conditional expectation have been chosen to reflect the properties of this sort of map.

Another property of the map L mentioned above is that its image is isomorphic to $\mathfrak{B}(Q(\mathcal{H}))$. The maps of this type are the natural compression of the operators on a Hilbert space to those on some closed linear subspace. Which makes conditional expectations of special interest to us in our discussion of hidden variables.

This way they (and thus conditional expectations are of interest to us in our discussion of hidden variables.

For unital C*-algebras the restriction of a conditional expectation to its own image is required to be the identity map. So if $\mathfrak{B}$ is a hidden theory of $\mathfrak{A}$ with respect to some conditional expectation L, the requirements of the definition of a hidden theory as given above would require $\mathfrak{A}$ to be a hidden theory of itself, which is quite a strong requirement. The requirements for a hidden theory should therefore be weakened slightly, to allow for situations in which this is not the case. The weakening is that for $A \in \mathfrak{A}$ the dispersion is not required to be reduced by L. Even with this weakening one may prove that no such hidden theory can exist if $\mathfrak{A}$ is noncommutative.

Theorem 6.28. *Let $\mathfrak{A}$ be a noncommutative C*-subalgebra of $\mathfrak{B}$ containing the unit of $\mathfrak{B}$. Then $\mathfrak{B}$ is not a hidden theory of $\mathfrak{A}$ with respect to any conditional expectation L.*

Proof. If $\mathfrak{A}$ is noncommutative, then it contains a *-subalgebra that is isomorphic to the algebra of complex 2×2 matrices, $\mathfrak{M}_2$. Let C and D be the elements of $\mathfrak{A}$, that correspond to

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ respectively.}$$

Let $\phi \in \mathcal{S}(\mathfrak{A})$ be a state whose restriction to $\mathfrak{M}_2$ is given by the density matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Now let A and B be arbitrary elements of $\mathfrak{A}$. We assume that $\mathfrak{B}$ is a hidden theory of $\mathfrak{A}$ with respect to a conditional expectation L. The trivial case $\mathfrak{A} = \mathfrak{B}$ is excluded on the basis that $\mathfrak{A}$ is noncommutative. Thus there is a $A' \in \mathfrak{B}$ that does not belong to $\mathfrak{A}$, such that $L(A') = A$. Because $\mathfrak{B}$ is a hidden theory of $\mathfrak{A}$, there is a probability measure μ that decomposes ϕ, i.e.,

$$\phi(LX) = \int \omega(X) d\mu(\omega). \tag{6.1}$$

The conditions for a hidden theory tell us that for all $\omega \in \operatorname{supp}(\mu)$.

$$\sigma_\phi(A) > \sigma_\omega(A'); \tag{6.2}$$
$$\sigma_\phi(B) \geq \sigma_\omega(B) \quad (LB = B), \tag{6.3}$$

so

$$\sigma_\phi(A)\sigma_\phi(B) > \sigma_\omega(A')\sigma_\omega(B). \tag{6.4}$$

By applying the Heisenberg uncertainty relation (Theorem 6.18) we get

$$\sqrt{\sigma_\phi(A)\sigma_\phi(B)} > \frac{1}{2}\left|\omega([A',B])\right| \quad \forall\omega\in\mathrm{supp}(\mu). \tag{6.5}$$

Integrating both sides we find

$$\begin{aligned}\sqrt{\sigma_\phi(A)\sigma_\phi(B)} &> \frac{1}{2}\int\left|\omega([A',B])\right|d\mu(\omega)\\ &\geq \frac{1}{2}\left|\int\omega([A',B])d\mu(\omega)\right|\\ &= \frac{1}{2}\left|\phi(L[A',B])\right|.\end{aligned} \tag{6.6}$$

From the definition of conditional expectation one may deduce that

$$L[B,X] = [LB,X] \quad \forall X\in\mathfrak{A}; B\in\mathfrak{B}. \tag{6.7}$$

Thus we find that for all $A,B\in\mathfrak{A}$

$$\sigma_\phi(A)\sigma_\phi(B) > \frac{1}{4}\left|\phi([A,B])\right|^2. \tag{6.8}$$

On the other hand, one may explicitly compute that

$$\sigma_\phi(C)\sigma_\phi(D) = \frac{1}{4}\left|\phi([C,D])\right|^2, \tag{6.9}$$

which is a contradiction. □

We thus find that one cannot reduce the dispersion of the states of a noncommutative C*-algebra by embedding it into a larger C*-algebra. Other similar results are possible for other types of L. For example one can prove that there are no hidden theories with regard to an L which preserves certain parts of the algebraic structure, if $\mathfrak{A}$ has no dispersion-free states. We will not go any further into this here, because it would lead us too far from our proper subject, Bell's inequality.

6.4 Bell's inequality in C*-algebras

We now turn to the main purpose of this chapter, describing in which way Bell's inequality can be given meaning in the formalism of C*-algebras and under what conditions it holds.

6.4.1 Bell's correlation and Bell's inequality

Two C*-subalgebras $\mathfrak{A}$ and $\mathfrak{B}$ of a C*-algebra $\mathfrak{C}$ are said to *commute* if $AB = BA$ for all $A\in\mathfrak{A}$ and $B\in\mathfrak{B}$. For two commuting C*-subalgebras one may define the notion of the Bell's correlation.

Definition 6.29. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-subalgebras of $\mathfrak{C}$ and let ϕ be a state on $\mathfrak{C}$.* ***Bell's correlation*** *$\beta(\phi, \mathfrak{A}, \mathfrak{B})$ is defined by*

$$\beta(\phi, \mathfrak{A}, \mathfrak{B}) := \frac{1}{2} \sup\{\phi(A_1(B_1 + B_2) + A_2(B_1 - B_2))\}, \tag{6.10}$$

where the supremum is taken over $A_1, A_2 \in \{A \in \mathfrak{A}_{\mathbb{R}} | \|A\| \leq 1\}$ and $B_1, B_2 \in \{B \in \mathfrak{B}_{\mathbb{R}} | \|B\| \leq 1\}$.

Note that the definition is symmetric in $\mathfrak{A}$ and $\mathfrak{B}$. Bell's inequality can now be formulated in the following way:

Definition 6.30.

Bell's inequality: $\beta(\phi, \mathfrak{A}, \mathfrak{B}) \leq 1.$

The following theorem gives us a number of conditions implying that Bell's inequality holds.

Theorem 6.31. *Bell's inequality $\beta(\phi, \mathfrak{A}, \mathfrak{B})$ holds in the following cases:*

(i) $\mathfrak{A}$ or $\mathfrak{B}$ is commutative.

(ii) ϕ is of the form $\phi = \sum_i \lambda_i \psi_i \chi_i$ with $\sum_i \lambda_i = 1$, $\psi_i \in \mathcal{S}(\mathfrak{A})$, and $\chi_i \in \mathcal{S}(\mathfrak{B})$.

(iii) The restriction of ϕ to either $\mathfrak{A}$ or $\mathfrak{B}$ is a pure state.

In the proof of (i) we will use the following lemma:

Lemma 6.32. *Let $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are commuting C*-subalgebras. If A is positive and B is self adjoint and satisfies $\|B\| \leq 1$, then $|\phi(AB)| \leq \phi(A)$.*

Proof of lemma. For any state ϕ and arbitrary X and Y, the following inequality holds:

$$\phi(X^* Y^* Y X) \leq \|Y\|^2 \phi(X^* X). \tag{6.11}$$

We use this inequality to prove the lemma

$$\begin{aligned}
|\phi(AB)| &= |\phi(AB_+) - \phi(AB_-)| && \text{by lemma 6.11} \\
&\leq \max\big(\phi(AB_+), \phi(AB_-)\big) && AB_\pm \text{ is positive (lemma 6.10).} \\
&= \max\big(\phi(a^* a b_+^* b_+), \phi(a^* a b_-^* b_-)\big) && \text{for some } a \in \mathfrak{A}, b_\pm \in \mathfrak{B} \\
&= \max\big(\phi(a^* b_+^* b_+ a), \phi(a^* b_-^* b_- a)\big) && \mathfrak{A} \text{ and } \mathfrak{B} \text{ commute.} \\
&\leq \max\big(\|b_+\|^2 \phi(a^* a), \|b_-\|^2 \phi(a^* a)\big) && \text{by equation (6.11)} \\
&= \max\big(\|B_+\| \phi(A), \|B_-\| \phi(A)\big) && \|B_\pm\| = \|b_\pm^* b_\pm\| = \|b_\pm\|^2 \\
&\leq \phi(A) && \|B_\pm\| \leq \|B\| \leq 1
\end{aligned}$$

□

Proof of (i). Because of the symmetrical role of $\mathfrak{A}$ and $\mathfrak{B}$ we may assume that $\mathfrak{A}$ is commutative. For any A with $\|A\| \leq 1$ the element $\mathbb{I} \pm A$ is positive. The following four elements are therefore positive (use lemma 6.10 and the commutativity of $\mathfrak{A}$):

$$\begin{aligned} A_{+,+} &= \tfrac{1}{4}(\mathbb{I}+A_1)(\mathbb{I}+A_2); \\ A_{+,-} &= \tfrac{1}{4}(\mathbb{I}+A_1)(\mathbb{I}-A_2); \\ A_{-,+} &= \tfrac{1}{4}(\mathbb{I}-A_1)(\mathbb{I}+A_2); \\ A_{-,-} &= \tfrac{1}{4}(\mathbb{I}-A_1)(\mathbb{I}-A_2). \end{aligned}$$

We can use these elements to rewrite $\beta(\phi,\mathfrak{A},\mathfrak{B})$ as follows:

$$\begin{aligned} \beta(\phi,\mathfrak{A},\mathfrak{B}) &= \tfrac{1}{2}\sup\{\phi(A_1(B_1+B_2)+A_2(B_1-B_2))\} \\ &= \tfrac{1}{2}\sup\{\phi(A_1B_1+A_1B_2+A_2B_1-A_2B_2)\} \\ &= \sup\{\phi(A_{+,+}B_1+A_{+,-}B_2-A_{-,+}B_2-A_{-,-}B_1)\} \\ &= \sup\{\phi(A_{+,+}B_1)+\phi(A_{+,-}B_2)-\phi(A_{-,+}B_2)-\phi(A_{-,-}B_1)\} \\ &\leq \sup\{|\phi(A_{+,+}B_1)|+|\phi(A_{+,-}B_2)|+|\phi(A_{-,+}B_2)|+|\phi(A_{-,-}B_1)|\} \\ &\leq \sup\{\phi(A_{+,+})+\phi(A_{+,-})+\phi(A_{-,+})+\phi(A_{-,-})\} \\ &= \sup\{\phi(A_{+,+}+A_{+,-}+A_{-,+}+A_{-,-})\} = \phi(\mathbb{I}) = 1. \end{aligned}$$

□

Proof of (ii). Because $\beta(\psi+\chi,\mathfrak{A},\mathfrak{B}) \leq \beta(\psi,\mathfrak{A},\mathfrak{B}) + \beta(\chi,\mathfrak{A},\mathfrak{B})$ it is enough to see that the statement is true when ϕ is a product state of some states $\psi \in \mathcal{S}(\mathfrak{A})$ and $\chi \in \mathcal{S}(\mathfrak{B})$. In that case we have

$$\phi(A_1(B_1+B_2)+A_2(B_1-B_2)) = \psi(A_1)\big(\chi(B_1)+\chi(B_2)\big)+\psi(A_2)\big(\chi(B_1)-\chi(B_2)\big). \tag{6.12}$$

The theory of functionals tells us that for any functional ω the following inequality holds:

$$|\omega(A)| \leq \|\omega\| \cdot \|A\|. \tag{6.13}$$

Because $\|\omega\| = 1$ for any state w and A_1, A_2, B_1, B_2 have a norm less than 1, we know that

$$\begin{aligned} |\psi(A_1)| &\leq 1 \qquad & |\chi(B_1)| &\leq 1 \\ |\psi(A_2)| &\leq 1 \qquad & |\chi(B_2)| &\leq 1. \end{aligned}$$

We therefore have

$$\begin{aligned} &\tfrac{1}{2}\left|\psi(A_1)(\chi(B_1)+\chi(B_2))+\psi(A_2)(\chi(B_1)-\chi(B_2))\right| \\ \leq &\tfrac{1}{2}\left|\phi(A_2)\right|\left|\chi(B_1)+\chi(B_2)\right|+\frac{1}{2}\left|\phi(A_2)\right|\left|\chi(B_1)-\chi(B_2)\right| \\ \leq &\tfrac{1}{2}\left|\chi(B_1)+\chi(B_2)\right|+\frac{1}{2}\left|\chi(B_1)-\chi(B_2)\right| \\ = &\max(\left|\chi(B_1)\right|,\left|\chi(B_2)\right|)\leq 1, \end{aligned}$$

and hence

$$\beta(\phi,\mathfrak{A},\mathfrak{B})\leq 1$$

. □

Proof of (iii). Again the symmetrical role of $\mathfrak{A}$ and $\mathfrak{B}$ allows us to assume that the restriction of ϕ to $\mathfrak{A}$ is a pure state. This is equivalent to the condition that if ω is a functional such that $|\omega(A)|\leq|\phi(A)|$ for all $A\in\mathfrak{A}$, then $\omega=\lambda\phi$ for some $-1\leq\lambda\leq 1$.
We consider the functional

$$\omega_B(A)\equiv\phi(AB)\quad A\in\mathfrak{A};B\in\mathfrak{B}. \tag{6.14}$$

One can prove that under the conditions above, the following inequality holds (the proof itself is quite technical and will not be given here):

$$|\omega_B(A)|=|\phi(AB)|\leq|\phi(A)|\quad\forall A\in\mathfrak{A}. \tag{6.15}$$

The purity of ϕ on $\mathfrak{A}$ now implies that for every $B\in\mathfrak{B}$ there exists a number $-1\leq\psi(B)\leq 1$ such that

$$\phi(AB)=\omega_B(A)=\psi(B)\phi(A)\quad A\in\mathfrak{A};B\in\mathfrak{B}. \tag{6.16}$$

We therefore find that ϕ is a product state, so the statement follows from (ii). □

Thus we see a number of conditions that imply that Bell's inequality holds. In particular, we see that Bell's inequality holds for any commutative C*-algebra, which we associate with classical theories.

6.4.2 Violation of Bell's inequality in quantum field theory

In standard quantum mechanics there is no notion of locality. It is therefore not really surprising that quantum mechanics predicts superluminal influences. Observables like spin do not really have a localization. In fact there is nothing in quantum mechanics itself, that suggests that you could not measure the spin of an electron whose (expected) location is on the other side of the galaxy.

It is therefore desirable to consider a theory that already incorporates notions like locality. Such a theory is algebraic quantum field theory (AQFT). In AQFT each region V of space-time is associated with a C*-algebra of observables. This *net of algebras* is required to satisfy certain axioms, which guarantee the Lorentz covariance of the theory, and certain properties like separability. In this section we will examine violations of Bell's inequality in AQFT. We will not go into the basics of AQFT much. This would go much to far for this paper. For a thorough discussion of the subject we refer to [22]. Here we will just mention the properties of AQFT as we require for our discussion.

We know by theorem 6.31 that if $\mathfrak{A}$ or $\mathfrak{B}$ is commutative, then Bell's inequality holds. But if neither $\mathfrak{A}$ nor $\mathfrak{B}$ is commutative, then it can be shown that, under certain additional conditions, there exist states for which Bell's inequality is violated.

Theorem 6.33. *Let $\mathfrak{M}$ and $\mathfrak{N}$ be a pair of commuting von Neumann algebras acting on a Hilbert space $\mathcal{H}$ such that for all $A \in \mathfrak{M}, B \in \mathfrak{N}$, $\|AB\| = \|A\|\|B\|$. Then, if neither $\mathfrak{M}$ nor $\mathfrak{N}$ is commutative, there exists a state ψ that violates Bell's inequality (as formulated in d efinition 6.30).*

The proof of the theorem uses the following lemma, which we will not prove here. For a proof see [38], lemma on page 184.

Lemma 6.34. *If $\mathfrak{M}$ is a noncommutative von Neumann algebra, then there exist two noncommutating projections P and Q such that $\|[P, Q]\| = \frac{1}{2}$.*

A projection P is a linear operator for which $P = P^* = P^2$ holds.

Proof of Theorem. If P is a projection, then $2P - \mathbb{I}$ is self-adjoint and has $\|2P - \mathbb{I}\| \leq 1$. Let $P_1, P_2 \in \mathfrak{M}, Q_1, Q_2 \in \mathfrak{N}$ be projections. Define

$$\begin{aligned} A_1 &\equiv 2P_1 - \mathbb{I}; \\ A_2 &\equiv 2P_2 - \mathbb{I}; \\ B_1 &\equiv 2Q_1 - \mathbb{I}; \\ B_2 &\equiv 2Q_2 - \mathbb{I}. \end{aligned}$$

If we introduce the notation

$$Z \equiv A_1(B_1 + B_2) + A_2(B_1 - B_2), \tag{6.17}$$

one can show by explicit calculation that

$$Z^2 = 4 + 16([P_1, P_2][Q_1, Q_2]). \tag{6.18}$$

There exists a state ϕ such that

$$|\phi([P_1, P_2][Q_1, Q_2])| = \|[P_1, P_2][Q_1, Q_2]\|. \tag{6.19}$$

We may assume that $\phi([P_1, P_2][Q_1, Q_2])$ is positive, because we can change the sign by exchanging P_1 and P_2. So there is a state such that

$$\phi(Z^2) = 4 + 16\|[P_1, P_2][Q_1, Q_2]\|. \tag{6.20}$$

We therefore have

$$4 + 16\|[P_1, P_2][Q_1, Q_2]\| = \phi(Z^2) \leq \|Z^2\| \leq 4 + 16\|[P_1, P_2][Q_1, Q_2]\|, \tag{6.21}$$

hence

$$\|Z\| = \sqrt{\|Z^2\|} = 2\sqrt{1 + 4\|[P_1, P_2][Q_1, Q_2]\|}. \tag{6.22}$$

So there must exist a state ψ such that

$$\begin{aligned} \frac{1}{2}\left|\psi(A_1(B_1 + B_2) + A_2(B_1 - B_2))\right| &= \sqrt{1 + 4\|[P_1, P_2][Q_1, Q_2]\|} \\ &= \sqrt{1 + 4\|[P_1, P_2]\|\|[Q_1, Q_2]\|}. \end{aligned} \tag{6.23}$$

We can now apply the mentioned lemma to find pairs of projections P_1, P_2 and Q_1, Q_2 such that

$$\begin{aligned} \|[P_1, P_2]\| &= \frac{1}{2}; \\ \|[Q_1, Q_2]\| &= \frac{1}{2}, \end{aligned}$$

and find that

$$\frac{1}{2}\left|\psi(A_1(B_1 + B_2) + A_2(B_1 - B_2))\right| = \sqrt{2}. \tag{6.24}$$

We see that ψ violates Bell's inequality. □

In AQFT the conditions of theorem 6.33 are met if the von Neumann algebras $\mathfrak{M}$ and $\mathfrak{N}$ are associated with two region of space-time V and W, that have a space-like separation.

6.4.3 Implications of Bell's inequality

We have seen that Bell's inequality holds in classical theories. We have also seen that it is violated in a non-classical theory like AQFT. But what exactly does this mean? The Bell correlation as introduced in this chapter up to this point is just a definition.

One thing we have seen is that if a state ϕ violates Bell's inequality, it cannot be a product state (theorem 6.31 (ii)). To see what this means, let us consider the following situation. Let V_1 and V_2 be space-like separated regions of space-time. Then, if ϕ_1 and ϕ_2 are the restrictions of ϕ to the C*-subalgebras $\mathfrak{A}(V_1)$ and $\mathfrak{A}(V_2)$, there are $A \in \mathfrak{A}(V_1)_{\mathbb{R}}$ and $B \in \mathfrak{A}(V_2)_{\mathbb{R}}$ for which $\phi(AB) \neq \phi_1(A)\phi_2(B) = \phi(A)\phi(B)$. So if the world is in state ϕ and we measure the observable connected with A in V_1, the expectation value will be $\phi(A)$. And if we measure the observable connected with B, the expectation value will be $\phi(B)$. But if we want to measure the product of these observables, which can be done by measuring A and B and then multiplying the results, the expectation value will not just be the product of the individual expectation values. And thus we find that there is some sort of correlation between the measurement outcomes of A and B. We conclude that violation of Bell's inequality in AQFT implies that there exist superluminal correlations of EPR-type.

There seems to be some disagreement in the literature on the question whether violation of Bell's inequality in AQFT actually excludes the possibility a common cause explanation for these superluminal correlations. Van Fraassen in [15] and Butterfield in [10] both come to the conclusion that common cause explanations are excluded. Redei disagrees with this conclusion ([38] chapter 12). His critique seems to focus on what is to be understood as a common cause. And in his understanding of the concept it seems that the question whether common causes are excluded is still open.

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8 A short guide through the literature

David Mermin has written a couple of (beautiful) popular articles which are very accessible. See, for example, [30, 31, 32]. They can also be found in Mermin's book "Boojums all the way throug", chapters 10-12. Another good starting point for studying Bell's inequality could be the original EPR article [14], which should be understandable even for the beginning student. Furthermore Bell's original article [4] can also be recommended to start with.

A great book on the philosophy and history of quantum mechanics is Max Jammer's book [25]. This book was our main source for chapter 2.

For a (deeper) study of Bell's inequality we recommend Jeffrey Bub's "Interpreting the Quantum World" [9]. Together with Maudlin's book [29], which focuses more on the relation between Bell's inequality and special relativity, we used it (besides other sources) for the chapters 3 and 4.

For the experimental side of the story we found the following two references useful: [1, 18].

A good starting point for studying C*-algebra's are (besides, of course, lectures at your university!) Klaas Landsman's lecture notes [28]. After having studied the first part thereof, one can move on to the interesting chapters of [38].

We also want to mention "Speakable and Unspeakable in Quantum Mechanics", [6]. In this book most of Bell's papers are collected and it is certainly worth having a look at it.